СРЕДЊЕКВАДРАТНА АПРОКСИМАЦИЈА

ЗАД: (25.06.2012.)

Наћи најбољу средњеквадратну апроксимацију функције $f(x) = \operatorname{sign} x$ на сегменту $[-1,1]$ полиномом трећег степена у Хилбертовом простору у коме ортогонални систем чине Лежандрови полиноми.

Решење:

$$\boxed{L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]}$$ (тежинска функ. $\rho(x) = 1$)

$n = 0,1,2,3$

$$L_0(x) = \frac{1}{1\cdot 1!}\cdot 1 = 1 \qquad \boxed{L_0(x) = 1}$$

$$L_1(x) = \frac{1}{2}\cdot\frac{d}{dx}(x^2-1) = \frac{1}{2}\cdot 2x = x, \quad \boxed{L_1(x) = x}, \quad \boxed{L_2(x) = \frac{3x^2-1}{2}}$$

$$L_2(x) = \frac{1}{4\cdot 2!}\frac{d^2}{dx^2}\left((x^2-1)^2\right) = \frac{1}{8}\cdot\frac{d}{dx}\left(2(x^2-1)\cdot 2x\right) = \frac{1}{2}\frac{d}{dx}(x^3-x) = \frac{1}{2}(3x^2-1)$$

$$L_3(x) = \frac{1}{8\cdot 3!}\frac{d^3}{dx^3}\left((x^2-1)^3\right) = \frac{1}{\cancel{48}_{8}}\cdot\frac{d^2}{dx^2}\left(\cancel{3}(x^2-1)^2\cdot \cancel{2}x\right) = \frac{1}{8}\frac{d}{dx}\left(2(x^2-1)\cdot 2x^2 + (x^2-1)^2\right)$$

$$= \frac{1}{8}\cdot\frac{d}{dx}\left[(x^2-1)(4x^2+x^2-1)\right] = \frac{1}{8}\cdot\frac{d}{dx}\left[(x^2-1)(5x^2-1)\right]$$

$$= \frac{1}{\cancel{8}_{4}}\left[\cancel{2}x(5x^2-1) + (x^2-1)\cdot \cancel{10}^{5}x\right] = \frac{1}{4}(5x^3 - x + 5x^3 - 5x) = \frac{1}{4}(10x^3 - 6x)$$

$$\boxed{L_3(x) = \frac{5x^3-3x}{2}}$$

$$\boxed{F(x) = \sum_{n=0}^{3} a_n \cdot L_n(x)}$$

$$\boxed{a_n = \frac{(f, L_n(x))}{(L_n, L_n)}}$$; скаларни производ у $L^2(-1,1)$ је дефинисан

са $(f,g) = \int_{-1}^{1} f(x)g(x)\,dx, \quad \left(f,g \in L^2(-1,1)\right)$

$$(L_n, L_n) = \|L_n\|^2 = \frac{2}{2n+1}, \quad n = 0, 1, 2, \dots$$

$$a_n = \frac{1}{\frac{2}{2n+1}} \int_{-1}^{1} f(x) L_n(x)\,dx = \frac{2n+1}{2} \int_{-1}^{1} f(x) L_n(x)\,dx$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} \operatorname{sign} x \, L_0(x)\,dx = \frac{1}{2} \int_{-1}^{1} \operatorname{sign} x \cdot 1 \,dx = 0$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} \operatorname{sign} x \cdot x \,dx = \frac{3}{2}$$

$$a_2 = \frac{5}{2} \int_{-1}^{1} \underbrace{\operatorname{sign} x \cdot \frac{3x^2-1}{2}}_{\text{нечетна}} \,dx = 0$$

$$a_3 = \frac{7}{2} \int_{-1}^{1} \operatorname{sign} x \cdot \frac{5x^3-3x}{2}\,dx = \frac{7}{4}\left[-\int_{-1}^{0} (5x^3-3x)\,dx + \int_{0}^{1} (5x^3-3x)\,dx\right]$$

$$= \frac{7}{4} \cdot \left(-\frac{1}{4} - \frac{1}{4}\right) = -\frac{7}{8}$$

$$F(x) = a_0 L_0(x) + a_1 L_1(x) + a_2 L_2(x) + a_3 L_3(x)$$

$$F(x) = \frac{3}{2} \cdot x - \frac{7}{8} \cdot \frac{5x^3-3x}{2} = \frac{3}{2}x - \frac{35}{16}x^3 + \frac{21}{16}x = \boxed{-\frac{35}{16}x^3 + \frac{45}{16}x}$$

4. (13.07.2012.)

Наћи најбољу средњеквадратну апроксимацију функције $f(x)=|x|$ на интервалу $[-1,1]$ полиномом четвртог степена у Хилбертовом простору у коме ортогонални систем функција чине Чебишевљеви полиноми.

Рјешење:

$$T_n(x) = (-1)^n 2^n \frac{n!}{(2n)!} \sqrt{1-x^2} \frac{d^n}{dx^n} (1-x^2)^{n-1/2} \quad , n=0,1,2,\dots$$

апроксимациона функција $\boxed{F(x) = \sum_{n=0}^{4} a_n \cdot T_n(x)}$

$T_n(x)$ Чебишевљеви полиноми, ортогонални на $[-1,1]$, са тежином $\boxed{p(x) = (1-x^2)^{-1/2}}$.

$$\boxed{a_n = \frac{(f,T_n)}{(T_n,T_n)}} \quad , n=0,1,2,3,4$$

скаларни производ у $L^2(-1,1)$ је дефинисан са

$$(f,g) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) g(x) dx \quad (f,g \in L^2(-1,1))$$

$$(T_n,T_n) = \|T_n\|^2 = \begin{cases} \pi, & n=0 \\ \frac{\pi}{2}, & n \neq 0 \end{cases}$$

$T_0(x) = 1$

$$a_0 = \frac{(f,T_0)}{\|T_0\|^2} = \frac{1}{\pi} \int_{-1}^{1} \underbrace{\frac{1}{\sqrt{1-x^2}} |x| \cdot T_0(x)}_{\text{парна фун.}} dx = \frac{2}{\pi} \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = \boxed{\frac{2}{\pi}}$$

$$T_1(x) = -2 \cdot \frac{1}{2} \sqrt{1-x^2} \cdot \frac{d}{dx}\left((1-x^2)^{1/2}\right) = -\sqrt{1-x^2} \cdot \frac{1}{\cancel{2}} (1-x^2)^{-1/2} (-\cancel{2}x)$$

$T_1(x) = x$

$$a_1 = \frac{(f,T_1)}{(T_1,T_1)} = \frac{1}{\frac{\pi}{2}} \int_{-1}^{1} \underbrace{\frac{1}{\sqrt{1-x^2}} |x| \cdot x}_{\text{непарне функ.}} dx = 0 \quad , \boxed{a_1 = 0}$$

$$T_2(x) = 4 \cdot \frac{2}{4!} \sqrt{1-x^2} \frac{d^2}{dx^2}\left((1-x^2)^{3/2}\right) = \frac{1}{\cancel{3}} \sqrt{1-x^2} \cdot \frac{d}{dx}\left(\frac{\cancel{3}}{\cancel{2}}(1-x^2)^{1/2} \cdot (-\cancel{2}x)\right)$$

$$= -\sqrt{1-x^2}\left(\frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)\cdot x + (1-x^2)^{1/2}\right] = -\sqrt{1-x^2}\left[(1-x^2)^{-1/2}(-x^2) + (1-x^2)^{1/2}\right]$$

$$= x^2 - (1-x^2) = 2x^2 - 1$$

$$\boxed{T_2(x) = 2x^2 - 1}$$

$$a_2 = \frac{(f, T_2)}{\|T_2\|^2} = \frac{2}{\pi} \int_{-1}^{1} \underbrace{\frac{1}{\sqrt{1-x^2}} |x| (2x^2-1)}_{\text{парне}} dx = \frac{4}{\pi} \int_0^1 \frac{x(2x^2-1)}{\sqrt{1-x^2}} dx$$

$$= \frac{4}{\pi}\left(2 \int_0^1 \frac{x \cdot x^2}{\sqrt{1-x^2}} dx - \int_0^1 \frac{dx}{\sqrt{1-x^2}}\right) = \frac{4}{\pi}\left(2 \cdot \frac{2}{3} - \frac{\pi}{2}\right) = \frac{4}{\pi}\left(\frac{4}{3} - \frac{\pi}{2}\right)$$

$$\boxed{a_2 = \frac{16}{3\pi} - 2}$$

$$T_3(x) = -\cancel{8} \cdot \frac{3!}{\cancel{6!}_{4\cdot 5\cdot \cancel{6}}} \sqrt{1-x^2} \frac{d^3}{dx^3}\left[(1-x^2)^{\frac{5}{2}}\right] = -\frac{1}{15}\sqrt{1-x^2} \frac{d^2}{dx^2}\left[\frac{5}{\cancel{2}}(1-x^2)^{3/2}(-\cancel{2}x)\right]$$

$$= \frac{1}{3}\sqrt{1-x^2} \frac{d}{dx}\left[\frac{3}{\cancel{2}}(1-x^2)^{1/2}(-\cancel{2}x)\cdot x + (1-x^2)^{3/2}\right] = \frac{1}{3}\sqrt{1-x^2}\left(-\frac{3}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)\cdot x^2 -\right.$$

$$= \frac{1}{3}\sqrt{1-x^2} \frac{d}{dx}\left[-3(1-x^2)^{1/2} \cdot x^2 + (1-x^2)^{3/2}\right] = \frac{1}{\cancel{3}}\sqrt{1-x^2}\left(\cancel{3}x^3(1-x^2)^{-1/2} - \cancel{6}x(1-x^2)^{1/2}\right.$$

$$- 6x(1-x^2)^{1/2} + \frac{3}{\cancel{2}}(1-x^2)^{1/2}(-\cancel{2}x)\Big]$$

$$\left. - \cancel{3}x(1-x^2)^{1/2}\right) = x^3 - 2x(1-x^2) - x(1-x^2) = x^3 - 2x + 2x^3 - x + x^3$$

$$\boxed{T_3(x) = 4x^3 - 3x} \qquad a_3 = \frac{2}{\pi} \int_{-1}^{1} \underbrace{\frac{1}{\sqrt{1-x^2}} |x| (4x^3 - 3x)}_{\text{непарне}} dx = 0 \qquad \boxed{a_3 = 0}$$

$$T_4(x) = 2^4 \frac{4!}{8!} \sqrt{1-x^2} \frac{d^4}{dx^4}\left[(1-x^2)^{\frac{7}{2}}\right]$$

$$= \cancel{16} \cdot \frac{1}{5\cdot \cancel{6}_3 \cdot 7 \cdot \cancel{8}}^{\frac{1}{15\cdot 7}} \sqrt{1-x^2} \frac{d^3}{dx^3}\left[\frac{7}{\cancel{2}}(1-x^2)^{\frac{5}{2}}(-\cancel{2}x)\right]$$

$$= -\frac{1}{15}\sqrt{1-x^2}\frac{d^3}{dx^3}\left[(1-x^2)^{\frac{5}{2}}\cdot x\right] = -\frac{1}{15}\sqrt{1-x^2}\frac{d^2}{dx^2}\left[\frac{5}{\cancel{2}}(1-x^2)^{3/2}(-\cancel{2}x^2) +\right.$$

$$\left. + (1-x^2)^{\frac{5}{2}}\right] = -\frac{1}{15}\sqrt{1-x^2}\frac{d^2}{dx^2}\left[-5x^2(1-x^2)^{3/2} + (1-x^2)^{5/2}\right]$$

$$= -\frac{1}{15}\sqrt{1-x^2}\frac{d}{dx}\left[-10x(1-x^2)^{3/2} - 5x^2\cdot\frac{3}{\cancel{2}}(1-x^2)^{1/2}\cdot(-\cancel{2}x) + \frac{5}{\cancel{2}}(1-x^2)^{3/2}(-\cancel{2}x)\right]$$

$$= -\frac{1}{\cancel{15}}\sqrt{1-x^2}\frac{d}{dx}\left[-\cancel{15}x(1-x^2)^{3/2} + \cancel{15}x^3(1-x^2)^{1/2}\right]$$

$$= \sqrt{1-x^2}\frac{d}{dx}\left[x(1-x^2)^{3/2} - x^3(1-x^2)^{1/2}\right]$$

$$= \sqrt{1-x^2}\left((1-x^2)^{3/2} + x\cdot\frac{3}{\cancel{2}}(1-x^2)^{1/2}(-\cancel{2}x) - 3x^2(1-x^2)^{1/2} - x^3\cdot\frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)\right)$$

$$= (1-x^2)^2 - 3x^2(1-x^2) - 3x^2(1-x^2) + x^4$$

$$= 1-2x^2+x^4-3x^2+3x^4-3x^2+3x^4+x^4 = 8x^4-8x^2+1$$

$$\boxed{T_4(x) = 8x^4-8x^2+1}$$

$$a_4 = \frac{(f,T_4)}{\|T_4\|^2} = \frac{2}{\pi}\underbrace{\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}|x|(8x^4-8x^2+1)}_{\text{парна}}dx = \frac{4}{\pi}\int_0^1 \frac{x(8x^4-8x^2+1)dx}{\sqrt{1-x^2}}$$

$$= \frac{4}{\pi}\left[8\underbrace{\int_0^1\frac{x^4\,x\,dx}{\sqrt{1-x^2}}}_{I_1 = \frac{8}{15}}dx - 8\underbrace{\int_0^1\frac{x^2\cdot x\,dx}{\sqrt{1-x^2}}}_{I_2 = \frac{2}{3}} + \underbrace{\overbrace{\int_0^1\frac{x\,dx}{\sqrt{1-x^2}}}^{1}}_{I_3}\right]$$

$$= \frac{4}{\pi}\left(8\cdot\frac{8}{15} - 8\cdot\frac{2}{3} + 1\right) = \frac{4}{\pi}\left(-\frac{1}{15}\right) = -\frac{4}{15\pi}$$

$$F(x) = a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + a_3 T_3(x) + a_4 T_4(x)$$

$$F(x) = \frac{2}{\pi} + \left(\frac{16}{3\pi} - 2\right)(2x^2 - 1) - \frac{4}{15\pi}(8x^4 - 8x^2 + 1)$$

$$= \frac{2}{\pi} + \left(\frac{32}{3\pi} - 4\right)x^2 - \frac{16}{3\pi} + 2 - \frac{32}{15\pi} x^4 + \frac{32}{15\pi} x^2 - \frac{4}{15\pi}$$

$$\boxed{F(x) = -\frac{32}{15\pi} x^4 + \left(\frac{64}{5\pi} - 4\right)x^2 - \frac{18}{5\pi} + 2}$$

Зад: Наћи најбољу средњеквадратну апроксимацију функције $f(x)=\arcsin x$ на интервалу $[-1,1]$ полиномом петог степена у Хилбертовом простору у коме ортогонални систем функција чине Чебишевљеви полиноми.

Решење: $F(x)=\sum_{n=0}^{5} a_n T_n(x)$

$$a_n=\frac{(f,T_n)}{(T_n,T_n)}, \quad (f,T_n)=\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}f(x)T_n(x)\,dx$$

$$(T_n,T_n)=\|T_n\|^2=\begin{cases}\pi, & n=0\\ \pi/2, & n\neq 0\end{cases}$$

$T_0(x)=1$, $T_1(x)=x$, $T_2(x)=2x^2-1$, $T_3(x)=4x^3-3x$

$T_4(x)=8x^4-8x^2+1$, $T_5(x)=16x^5-20x^3+5x$

$$a_0=\frac{(f,T_0)}{\|T_0\|^2}=\frac{1}{\pi}\int_{-1}^{1}\underbrace{\frac{1}{\sqrt{1-x^2}}\arcsin x}_{\text{непарна}}\,dx=0$$

$$a_1=\frac{2}{\pi}\int_{-1}^{1}\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx=\frac{4}{\pi}\int_{0}^{1}\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx=\left|\begin{matrix}u=\arcsin x, & du=\frac{dx}{\sqrt{1-x^2}}\\ v=\int\frac{x\,dx}{\sqrt{1-x^2}}=-\sqrt{1-x^2}\end{matrix}\right|$$

$$=\frac{4}{\pi}\left(\underbrace{-\arcsin x\cdot\sqrt{1-x^2}\Big|_0^1}_{0}+\int_0^1 dx\right)=\frac{4}{\pi}\left(0+x\Big|_0^1\right)=\boxed{\frac{4}{\pi}}$$

$$a_2=\frac{(f,T_2)}{\|T_2\|^2}=\frac{2}{\pi}\int_{-1}^{1}\frac{2x^2-1}{\sqrt{1-x^2}}\arcsin x\,dx=\boxed{0}$$

$$a_3=\frac{(f,T_3)}{\|T_3\|^2}=\frac{2}{\pi}\int_{-1}^{1}\frac{4x^3-3x}{\sqrt{1-x^2}}\arcsin x\,dx=\frac{4}{\pi}\int_{0}^{1}\frac{4x^3-3x}{\sqrt{1-x^2}}\arcsin x\,dx$$

$$=\frac{4}{\pi}\left[4\int_0^1\frac{x^3\arcsin x}{\sqrt{1-x^2}}\,dx-3\underbrace{\int_0^1\frac{x\arcsin x}{\sqrt{1-x^2}}\,dx}_{=1}\right]$$

$$\int_0^1 \frac{x^3 \arcsin x}{\sqrt{1-x^2}}dx = \left| \begin{array}{l} u = x^2 \\ v = \int \frac{x \arcsin x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}\cdot \arcsin x + x \end{array} \right|$$

$$= x^2(-\arcsin x \cdot \sqrt{1-x^2} + x)\Big|_0^1 - 2\int_0^1 x(-\sqrt{1-x^2}\cdot \arcsin x + x)\,dx$$

$$= 1 + 2\int_0^1 x\sqrt{1-x^2}\arcsin x\,dx - 2\int_0^1 x^2 dx = 1 - \frac{2}{3} + 2\int_0^1 x\sqrt{1-x^2}\cdot \arcsin x\,dx$$

$$\left| \begin{array}{l} u = \arcsin x \qquad v = \int x\sqrt{1-x^2}dx = -\frac{\sqrt{(1-x^2)^3}}{3} \\ du = \frac{dx}{\sqrt{1-x^2}} \end{array} \right| = \frac{1}{3} + 2\Big(\cancel{-\frac{\sqrt{(1-x^2)^3}}{3}\arcsin x\Big|_0^1}_{0} +$$

$$+\frac{1}{3}\int_0^1 (1-x^2)dx\Big) = \frac{1}{3} + 2/3\left(x\Big|_0^1 - \frac{x^3}{3}\Big|_0^1\right) = \frac{1}{3} + \frac{2}{3}\left(1-\frac{1}{3}\right) = \frac{1}{3} + \frac{4}{9} = \frac{7}{9}$$

$$a_3 = \frac{4}{\pi}\left(4\cdot\frac{7}{9} - 3\right) = \boxed{\frac{4}{9\pi}}$$

$$a_4 = \frac{2}{\pi}\int_{-1}^1 \frac{8x^4 - 8x^2 + 1}{\sqrt{1-x^2}}\arcsin x\,dx = \boxed{0}$$

$$a_5 = \frac{2}{\pi}\int_{-1}^1 \frac{16x^5 - 20x^3 + 5x}{\sqrt{1-x^2}}\arcsin x\,dx = \frac{4}{\pi}\int_0^1 \frac{16x^5 - 20x^3 + 5x}{\sqrt{1-x^2}}\arcsin x\,dx$$

$$= \frac{4}{\pi}\left(16\int_0^1 \frac{x^5 \arcsin x\,dx}{\sqrt{1-x^2}} - 20\underbrace{\int_0^1 \frac{x^3 \arcsin x\,dx}{\sqrt{1-x^2}}}_{\frac{7}{9}} + 5\underbrace{\int_0^1 \frac{x \arcsin x}{\sqrt{1-x^2}}dx}_{1}\right)$$

$$\int_0^1 \frac{x^5 \arcsin x\,dx}{\sqrt{1-x^2}} = \left| \begin{array}{l} u = x^4, \quad du = 4x^3 \\ v = \int \frac{x\arcsin x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}\arcsin x + x \end{array} \right|$$

$$= \cancel{x^4(-\sqrt{1-x^2}\arcsin x + x)}_{0}\Big|_0^1 - 4\int_0^1 x^3(-\sqrt{1-x^2}\arcsin x + x)\,dx$$

$$= 4\int_0^1 x^3\sqrt{1-x^2}\arcsin x\,dx - 4\int_0^1 x^4 dx = -\frac{4}{5} + 4\int_0^1 x\cdot x^2\sqrt{1-x^2}\arcsin x\,dx$$

$$= -\frac{4}{5} + 4\Big[\arcsin x\cdot\Big(\cancel{\frac{\sqrt{(1-x^2)^5}}{5} - \frac{\sqrt{(1-x^2)^3}}{3}}_{0}\Big)\Big|_0^1 - \int_0^1\left(\frac{1}{5}\sqrt{(1-x^2)^5} - \frac{1}{3}\sqrt{(1-x^2)^3}\right)\frac{dx}{\sqrt{1-x^2}}\Big)$$

$$= -\frac{4}{5} - 4\left(\int_0^1 \frac{1}{5}(1-x^2)^2 dx - \int_0^1 \frac{1}{3}(1-x^2) dx\right) = -\frac{4}{5} - 4\left(\frac{1}{5}\int_0^1 (1-2x^2+x^4) dx - \frac{1}{3}\int_0^1 (1-x^2) dx\right.$$

$$= -\frac{4}{5} - 4\left(\frac{1}{5} \cdot \frac{8}{15} - \frac{2}{9}\right) = -\frac{4}{5} - 4 \cdot \left(\frac{-26}{225}\right) = \frac{-76}{225}$$

$$a_5 = \frac{4}{\pi}\left(16 \cdot \left(\frac{-76}{225}\right) - 20 \cdot \frac{7}{9} + 5\right) = \frac{4}{\pi} \cdot \frac{-32319}{225 \cdot 9} = -\frac{4}{\pi} \cdot \frac{32319}{2025}$$

$$F(x) = \frac{4}{\pi} x + \frac{4}{9\pi}(4x^3 - 3x) - \frac{4}{\pi} \cdot \frac{32319}{2025} \cdot (16x^5 - 20x^3 + 5x)$$