КРИВОЛИНИЈСКИ ИНТЕГРАЛ ПО ЛУКУ

$$\int_L f(x,y,z)\,dl \qquad , \quad dl = \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

ЗАД 1: Израчунати криволинијски интеграл $I=\int_C y\,ds$ гдје је C лемниската $(x^2+y^2)^2=a^2(x^2-y^2)$.

Рјешење:

пређимо на поларни координатни систем

$x=\rho\cos\varphi \qquad \varphi\in[0,2\pi]$

$y=\rho\sin\varphi \qquad \rho\in(0,+\infty)$

$$\Rightarrow \quad \rho^4=a^2\rho^2\cos2\varphi$$

$$\rho^2=a^2\cos2\varphi \quad \Rightarrow \quad \varphi\in\left[-\frac{\pi}{4}+k\pi,\ \frac{\pi}{4}+k\pi\right]$$

$$\boxed{\rho=a\sqrt{\cos2\varphi}}$$

$$dl=\sqrt{\rho^2+(\rho'_\varphi)^2}\,d\varphi \quad , \quad \rho'_\varphi=-\frac{a\sin2\varphi}{\sqrt{\cos2\varphi}}$$

$$dl=\sqrt{a^2\cos2\varphi+\frac{a^2\sin^22\varphi}{\cos2\varphi}}\,d\varphi=\frac{a\,d\varphi}{\sqrt{\cos2\varphi}}$$

$$I=4\int_0^{\pi/4}a\cancel{\sqrt{\cos2\varphi}}\cdot\sin\varphi\cdot\frac{a\,d\varphi}{\cancel{\sqrt{\cos2\varphi}}}=4a^2\int_0^{\pi/4}\sin\varphi\,d\varphi=2a^2(2-\sqrt2)$$

ЗАД 2: Израчунати криволинијски интеграл

$$I=\oint_C\left(\sqrt[3]{x^4}+\sqrt[3]{y^4}\right)ds \quad , \text{ ако је } C \text{ астроида}$$

$$\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{9}.$$

Рјешење: параметризација: $x=3\cos^3t \quad , \quad y=3\sin^3t \quad , \quad t\in[0,2\pi]$

$$x'_t=9\cos^2t(-\sin t)dt \quad , \quad y'_t=9\sin^2t\cdot\cos t\,dt$$

$$ds=\sqrt{(x')^2+(y')^2}\,dt=\sqrt{9^2\cos^4t\sin^2t+9^2\sin^4t\cos^2t}\,dt=9\sqrt{\sin^2t\cos^2t(\cos^2t+\sin^2t)}\,dt$$

$$ds=9|\sin t\cos t|\,dt$$

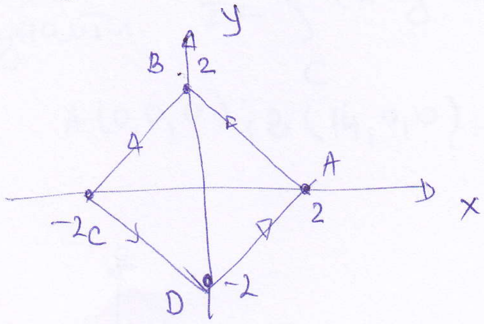
$$I = 9\cdot 3^{4/3}\int_0^{2\pi}(\cos^4 t+\sin^4 t)|\sin t\cos t|\,dt = 4\cdot 9\cdot 3^{4/3}\int_0^{\pi/2}(\cos^4 t+\sin^4 t)\sin t\cos t\,dt$$

$$= 36\cdot 3\sqrt[3]{3}\left(\int_0^{\pi/2}\cos^5 t\sin t\,dt+\int_0^{\pi/2}\sin^5 t\cos t\,dt\right) = \underline{36\sqrt[3]{3}}$$

<u>ЗАД 3:</u> Израчунати криволинијски интеграл $\int_C x^2 y\,dl$ ако је

$C: |x|+|y|=2.$

<u>РЈЕШЕЊЕ:</u>



$$\int_C = \int_{\overline{AB}} + \int_{\overline{BC}} + \int_{\overline{CD}} + \int_{\overline{DA}}$$

$\int_{\overline{AB}}$: $y = 2-x$, $x\in[0,2]$ $\qquad dl = \sqrt{1+1}\,dx = \sqrt{2}\,dx$

$dy = -dx$, $dx = dx$

$$\int_{\overline{AB}} = \int_0^2 x^2(2-x)\sqrt{2}\,dx = \sqrt{2}\left(2\frac{x^3}{3}\Big|_0^2 - \frac{x^4}{4}\Big|_0^2\right) = \frac{4}{3}\sqrt{2}$$

$\overline{BC}$: $y = x+2$, $x\in[0,-2]$ $\qquad dl = \sqrt{2}\,dx$

$dy = dx$

$$\int_{\overline{BC}} = \sqrt{2}\int_0^{-2} x^2(x+2)\,dx = \sqrt{2}\left(\frac{x^4}{4}+\frac{2x^3}{3}\right)\Big|_0^{-2} = -\frac{4\sqrt{2}}{3}$$

$\overline{CD}$: $y = -x-2$, $x\in[-2,0]$ $\qquad dl = \sqrt{2}\,dx$

$dy = -dx$

$$\int_{\overline{CD}} = \sqrt{2}\int_{-2}^{0} x^2(-x-2)\,dx = -\sqrt{2}\left(\frac{x^4}{4}+2\frac{x^3}{3}\right)\Big|_{-2}^{0} = -\frac{4\sqrt{2}}{3}$$

$\overline{DA}$: $y = x-2$, $x\in[0,2]$ $\quad dl=\sqrt{2}\,dx$

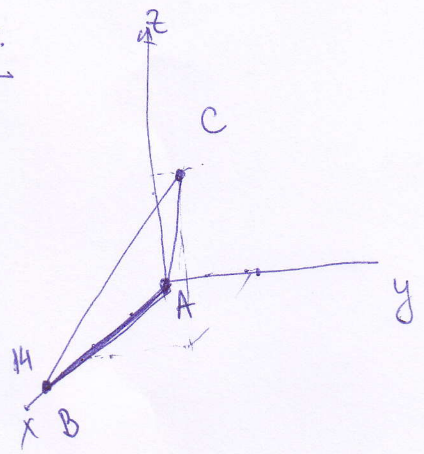
$dy=dx$

$$\int_{DA} = \sqrt{2}\int_0^2 x^2(x-2)\,dx = \sqrt{2}\left(\frac{x^4}{4}-\frac{2x^3}{3}\right)\Big|_0^2 = -\frac{4\sqrt{2}}{3}$$

$$I = \frac{4}{3}\sqrt{2} - 3\cdot\frac{4\sqrt{2}}{3} = \boxed{-\frac{8}{3}\sqrt{2}}$$

Зад: Израчунати $I=\int_C (x-y+2z)\,ds$, где је C контура троугла $A(0,0,0)$; $B(14,0,0)$, $C\left(9, \frac{36}{5}, \frac{48}{5}\right)$.

Решење:



$$I = \int_{AB}+\int_{BC}+\int_{CA}$$

$\overline{AB}$: $x=x$, $y=0$, $z=0$, $\quad ds=\sqrt{1+0+0}\,dx = dx$

$x\in[0,14]$

$$\int_{AB}(x-y+2z)\,ds = \int_0^{14} x\,dx = 98$$

$\overline{BC}$: $\dfrac{x-14}{-5} = \dfrac{y}{\frac{36}{5}} = \dfrac{z}{\frac{48}{5}} = t \Rightarrow$ $\quad x=-5t+14$, $\quad y=\frac{36}{5}t$, $\quad z=\frac{48}{5}t$ $\quad t\in[0,1]$

$$ds=\sqrt{25+\left(\frac{36}{5}\right)^2+\left(\frac{48}{5}\right)^2}\,dt = 13\,dt$$

$$\int_{BC} = \int_0^1\left(-5t+14-\frac{36}{5}t+\frac{96}{5}t\right)13\,dt = \frac{455}{2}$$

$\overline{CA}: \quad \frac{x}{9} = \frac{y}{\frac{36}{5}} = \frac{z}{\frac{48}{5}} = t \quad , t \in [0,1]$

$x = 9t, \; y = \frac{36}{5}t, \; z = \frac{48}{5}t ,$

$ds = \sqrt{9^2 + \left(\frac{36}{5}\right)^2 + \left(\frac{48}{5}\right)^2}\, dt = 15dt$

$\int_{CA} = \int_0^1 \left(9t - \frac{36}{5}t + \frac{96}{5}t\right) 15dt = 15\int_0^1 21t\, dt = \frac{315}{2}$

$I = \int_{AB} + \int_{BC} + \int_{CA} = 98 + \frac{455}{2} + \frac{315}{2} = 483$

Д: (22.06.2012.)

Израчунати криволинијски интеграл

$$\oint_L e^x(1-\cos y)\,dx + e^x(\sin y - y)\,dy$$

ако је L крива која ограничава област $0 \le x \le \pi$, $0 \le y \le \sin x$.

Рјешење: На основу Грин-Гаусове теореме добијамо

$$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \iint_D [(\sin y - y)e^x - e^x \sin y]\,dxdy =$$

$$= -\iint_D y e^x\,dxdy = -\int_0^\pi dx \int_0^{\sin x} e^x y\,dy = -\int_0^\pi e^x \cdot \frac{y^2}{2}\Big|_0^{\sin x} dx$$

$$= -\frac{1}{2}\int_0^\pi e^x \frac{1-\cos 2x}{2}\,dx = -\frac{1}{4}\left[\int_0^\pi e^x dx - \int_0^\pi e^x \cos 2x\,dx\right]$$

$$= -\frac{1}{4}\left[e^x\Big|_0^\pi - \frac{4}{5}\cdot\frac{e^x}{2}\left(\sin 2x + \frac{\cos 2x}{2}\right)\Big|_0^\pi\right]$$

$$= -\frac{1}{4}\left[e^\pi - 1 - \frac{e^x}{5}(2\sin 2x + \cos 2x)\Big|_0^\pi\right] = \ldots = \boxed{\frac{1}{5}(1-e^\pi)}$$