ДВОСТРУКИ ИНТЕГРАЛИ
(ИСПИТНИ ЗАДАЦИ)

ЗАД 1. (16.04.2013.)

Нека је $f(0,0)=0$ и $f(x,y)=\frac{y^2-x^2}{(x^2+y^2)^2}$, ако је $(x,y)\neq(0,0)$.

Показати да је $\int_0^1 dy\int_0^1 f(x,y)dx=\frac{\pi}{4}$ и $\int_0^1 dx\int_0^1 f(x,y)dy=-\frac{\pi}{4}$

Рјешење: $A=\int_0^1 dy\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}dx=\int_0^1 dy\int_0^1\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}dx=$

$$=\int_0^1 dy\left(\int_0^1\frac{x^2+y^2}{(x^2+y^2)^2}dx-2\int_0^1\frac{x^2dx}{(x^2+y^2)^2}\right)=\int_0^1 dy\left(\underbrace{\int_0^1\frac{dx}{x^2+y^2}}_{I_1}-2\underbrace{\int_0^1\frac{x\cdot xdx}{(x^2+y^2)^2}}_{I_2}\right)$$

$$I_1=\int_0^1\frac{dx}{y^2\left(1+\left(\frac{x}{y}\right)^2\right)}=\frac{1}{y^2}\cdot y\operatorname{arctg}\frac{x}{y}\Big|_0^1=\frac{1}{y}\operatorname{arctg}\frac{1}{y}$$

$$I_2=\left|\begin{array}{ll}u=x, & du=dx\\ dv=\frac{xdx}{(x^2+y^2)^2}, & v=\int\frac{xdx}{(x^2+y^2)^2}=\frac{1}{2}\left(-\frac{1}{x^2+y^2}\right)\end{array}\right|$$

$$=-\frac{x}{2(x^2+y^2)}\Big|_0^1+\frac{1}{2}\int_0^1\frac{dx}{x^2+y^2}=-\frac{1}{2(1+y^2)}+\frac{1}{2}\cdot\frac{1}{y}\operatorname{arctg}\frac{1}{y}$$

$$A=\int_0^1\frac{1}{y}\operatorname{arctg}\frac{1}{y}dy-2\left(-\int_0^1\frac{1}{2(1+y^2)}dy+\frac{1}{2}\int_0^1\frac{1}{y}\operatorname{arctg}\frac{1}{y}dy\right)$$

$$=\cancel{\int_0^1\frac{1}{y}\operatorname{arctg}\frac{1}{y}dy}+\int_0^1\frac{dy}{1+y^2}-\cancel{\int_0^1\frac{1}{y}\operatorname{arctg}\frac{1}{y}dy}=\operatorname{arctg}y\Big|_0^1=\frac{\pi}{4}$$

$$B=\int_0^1 dx\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}dy = -\int_0^1 dx\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy$$

$$=-\int_0^1 dx\int_0^1 \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = -\int_0^1 dx\left(\int_0^1 \frac{x^2+y^2}{(x^2+y^2)^2}dy - 2\int_0^1 \frac{y^2}{(x^2+y^2)^2}dy\right)$$

(аналогно као код А) и добијамо

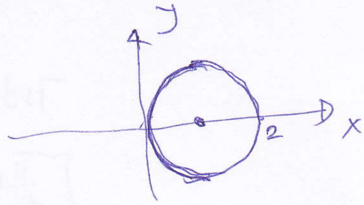
$$B=-\frac{\pi}{4}$$

ЗАД 2: (5.07.2013.)

Израчунати двојни интеграл $\iint_D (x^2+y^2)dxdy$ ако је D круг $x^2+y^2\le 2x$.

Решење: $D: x^2+y^2\le 2x$

$(x-1)^2+y^2\le 1$



прелазимо на поларни координатни систем

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$\rho^2=2\rho\cos\varphi$

$\rho=2\cos\varphi \Rightarrow \rho\in(0,2\cos\varphi)$

$\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $|J|=\rho$

$$I=\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{2\cos\varphi}\rho^2\cdot\rho\, d\rho = \int_{-\pi/2}^{\pi/2}\frac{\rho^4}{4}\Big|_0^{2\cos\varphi}d\varphi = 4\int_{-\pi/2}^{\pi/2}\cos^4\varphi\, d\varphi$$

$$=4\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\varphi}{2}\right)^2 d\varphi = \int_{-\pi/2}^{\pi/2}\left(1+2\cos 2\varphi+\frac{1+\cos 4\varphi}{2}\right)d\varphi$$

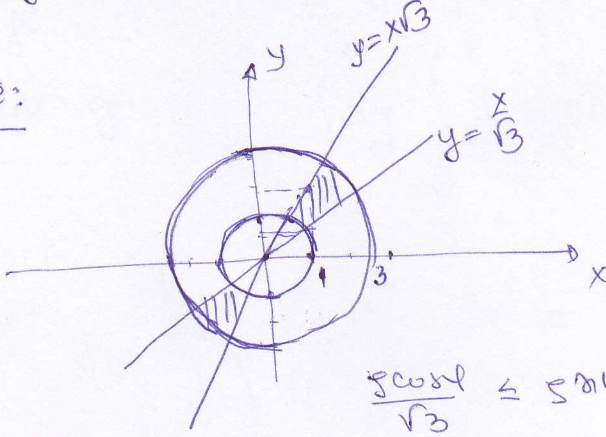
$$=\left(\varphi+\sin 2\varphi+\frac{1}{2}\varphi+\frac{1}{8}\sin 4\varphi\right)\Big|_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

Зад 3: (4.05.2015.)

Израчунати двојни интеграл $\iint\limits_D \operatorname{arctg}\frac{y}{x}\,dxdy$ гдје је

$$D=\left\{(x,y): 1\le x^2+y^2\le 9 \ \land \ \frac{x}{\sqrt3}\le y\le x\sqrt3\right\}.$$

Рјешење:



$$x=\rho\cos\varphi$$
$$y=\rho\sin\varphi$$
$$|J|=\rho$$
$$1\le\rho^2\le 9 \Rightarrow \rho\in[1,3]$$

$$\frac{\rho\cos\varphi}{\sqrt3}\le\rho\sin\varphi\le\sqrt3\,\rho\cos\varphi \quad \Big| : \rho\cos\varphi$$

$$\frac{1}{\sqrt3}\le \operatorname{tg}\varphi\le\sqrt3$$

$$\frac{\pi}{6}+k\pi\le\varphi\le\frac{\pi}{3}+k\pi$$

$$\varphi\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]\cup\left[\frac{7\pi}{6},\frac{4\pi}{3}\right]$$

$$I=\int_{\pi/6}^{\pi/3}d\varphi\int_1^3\operatorname{arctg}\frac{\rho\sin\varphi}{\rho\cos\varphi}\,\rho\,d\rho+\int_{7\pi/6}^{4\pi/3}d\varphi\int_1^3\underbrace{\operatorname{arctg}(\operatorname{tg}\varphi)}_{\varphi}\,\rho\,d\rho$$

$$=\int_{\pi/6}^{\pi/3}d\varphi\int_1^3\operatorname{arctg}(\operatorname{tg}\varphi)\,\rho\,d\rho+\int_{7\pi/6}^{4\pi/3}\varphi\,d\varphi\int_1^3\rho\,d\rho$$

$$=\frac{\rho^2}{2}\Big|_1^3\left[\frac{\varphi^2}{2}\Big|_{\pi/6}^{\pi/3}+\frac{\varphi^2}{2}\Big|_{7\pi/6}^{4\pi/3}\right]=\frac14(9-1)\left(\frac{\pi^2}{9}-\frac{\pi^2}{36}+\frac{16\pi^2}{9}-\frac{49\pi^2}{36}\right)$$

$$=2\pi^2\left(\frac{17}{9}-\frac{50}{36}\right)=\pi^2$$