ЗАД: (23.10.2014.) Одредити опште рјешење диференцијалне једначине

$$x(1-x^2)y' + (2x^2-1)y - x^3y^3 = 0$$

$y' + p(x)y = g(x)y^{\alpha}$, $\alpha \neq 0,1$ – БЕРНУЛИЈЕВА диф. јед., смјена $z = y^{1-\alpha}$

Рјешење:

$$y' + \frac{2x^2-1}{x(1-x^2)}y = \frac{x^3}{\cancel{x}(1-x^2)}y^3$$ – БЕРНУЛИЈЕВА диф. јед.

$\alpha = 3$, смјена $z = y^{1-\alpha} = y^{-2}$

$$z' = -2y^{-3}\cdot y' \Rightarrow y' = -\frac{1}{2}y^3 z'$$

$$-\frac{1}{2}y^3 z' + \frac{2x^2-1}{x(1-x^2)}y = \frac{x^2}{1-x^2}y^3 \quad \Big/ \cdot \frac{-2}{y^3}$$

$$z' + \frac{2(1-2x^2)}{x(1-x^2)}z = \frac{-2x^2}{1-x^2}$$ – лин. диф. јед.

$$z(x) = e^{-\int \frac{2(1-2x^2)}{x(1-x^2)}dx}\left(C + \int \frac{-2x^2}{1-x^2} e^{\overbrace{\int \frac{2(1-2x^2)}{x(1-x^2)}dx}^{I_1}} dx\right)$$

$$I_1 = 2\int \frac{2x^2-1}{x(1-x^2)}dx$$

$$\frac{2x^2-1}{x(1-x^2)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$$2x^2-1 = A(1-x^2) + Bx(1+x) + Cx(1-x)$$

$$\left.\begin{aligned} -A+B-C &= 2 \\ B+C &= 0 \\ A &= -1 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} A &= -1 \\ B &= 1/2 \\ C &= -1/2 \end{aligned}}$$

$$I_1 = 2\left[-\int\frac{dx}{x} + \frac{1}{2}\int\frac{dx}{1-x} - \frac{1}{2}\int\frac{dx}{1+x}\right] = 2\left[-\ln|x| - \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x|\right]$$

$$= -\ln x^2(1-x^2) = \ln[x^2(1-x^2)]^{-1}$$

$$z(x) = e^{\ln[x^2(1-x^2)]^{-1}}\left(C + \int\left(\frac{-2x^2}{1-x^2}\right)e^{\ln x^2(1-x^2)}dx\right]$$

$$z(x) = \frac{1}{x^2(1-x^2)}\left(c + \int \frac{-2x^2}{\cancel{1-x^2}} \cdot x^2 \cancel{(1-x^2)}\, dx\right)$$

$$= \frac{1}{x^2(1-x^2)}\left(c - 2\int x^4 dx\right) = \frac{1}{x^2(1-x^2)}\left(c - 2 \cdot \frac{x^5}{5}\right)$$

$$z(x) = \frac{1}{x^2(1-x^2)}\left(c - \frac{2}{5}x^5\right)$$

$$\frac{1}{y^2} = \frac{c - \frac{2}{5}x^5}{x^2(1-x^2)} \Rightarrow y^2\left(c - \frac{2}{5}x^5\right) = x^2(1-x^2)$$

$$\boxed{y^2 = \frac{5x^2(1-x^2)}{5c - 2x^5}}$$

ЗАД: (4.05.2015.)

Ријешити диференцијалну једначину

$$4xy' + y + 4xe^{\sqrt{x}}y^3 = 0 \quad \text{ако је} \quad y(1) = 1.$$

Рјешење: $y' + \frac{1}{4x}y = -e^{\sqrt{x}}y^3$ – БЕРНУЛИЈЕВА ДИФ. ЈЕД.

$\alpha = 3$, $\quad z = y^{1-\alpha} = y^{-2}$, $\quad z' = -2y^{-3}\cdot y'$

$$\boxed{y' = -\frac{z'}{2}y^3}$$

$$-\frac{z'}{2}y^3 + \frac{1}{4x}y = -e^{\sqrt{x}}y^3 \quad \Big/ \cdot \left(-\frac{2}{y^3}\right)$$

$$z' - \frac{1}{2x}\cdot\frac{1}{y^2} = 2e^{\sqrt{x}} \Rightarrow \underline{z' - \frac{1}{2x}z = 2e^{\sqrt{x}}}$$ ЛИН. ДИФ. ЈЕД.

$$z(x) = e^{\int\frac{dx}{2x}}\left(c + 2\int e^{\sqrt{x}}\cdot e^{-\int\frac{dx}{2x}}dx\right) = e^{\frac{1}{2}\ln|x|}\left(c + 2\int e^{\sqrt{x}}e^{-\frac{1}{2}\ln|x|}dx\right)$$

$$= \sqrt{x}\left(c + 2\int e^{\sqrt{x}}\cdot\frac{dx}{\sqrt{x}}\right) \quad \left|\begin{array}{l}\sqrt{x} = t\\ \frac{dx}{2\sqrt{x}} = dt\\ \frac{dx}{\sqrt{x}} = 2dt\end{array}\right| = \sqrt{x}\left(c + 2\cdot 2\int e^t dt\right)$$

$$\boxed{z(x) = \sqrt{x}\left(c + 4e^{\sqrt{x}}\right)}$$

$$\frac{1}{y^2} = \sqrt{x}\left(c + 4e^{\sqrt{x}}\right) \Rightarrow y^2 = \frac{1}{\sqrt{x}\left(c + 4e^{\sqrt{x}}\right)} \Rightarrow y = \pm\frac{1}{\sqrt{\sqrt{x}\left(c + 4e^{\sqrt{x}}\right)}}$$

$$y(1) = 1 \Rightarrow 1 = \frac{1}{\sqrt{c + 4e}} \Rightarrow \sqrt{c+4e} = 1 \; \Big/^2 \quad c + 4e = 1 \Rightarrow \boxed{c = 1 - 4e}$$

$$y = \frac{1}{\sqrt{\sqrt{x}\left(1 - 4e + 4e^{\sqrt{x}}\right)}}$$

ЗАД: (22.06.2012.) Показати да диференцијална једначина $2x^2y' = x^2y^2+1$ има партикуларно рјешење облика $y_1 = a + \frac{b}{x}$, па затим одредити опште рјешење.

Рјешење: $y_1' = -\frac{b}{x^2}$

$$2x^2\left(-\frac{b}{x^2}\right) = x^2y^2+1 \Rightarrow -2b = x^2\left(a+\frac{b}{x}\right)^2+1$$

$$-2b = x^2\left(a^2+\frac{2ab}{x}+\frac{b^2}{x^2}\right)+1$$

$$-2b = a^2x^2+2abx+b^2+1$$

$$a^2x^2+2abx+b^2+2b+1=0$$

$$\boxed{a=0} \wedge 2ab=0 \wedge b^2+2b+1=0$$

$$(b+1)^2=0$$

$$\boxed{b=-1}$$

$$\boxed{y_1 = -\frac{1}{x}}$$

$$2x^2y' = x^2y^2+1 \quad /:2x^2$$

$$y' = \frac{1}{2}y^2 + \frac{1}{2x^2}$$ – Рикатијева диф. јед.

$$\boxed{\begin{gathered} y' + p(x)y = g(x) + r(x)y^2 \\ g(x), r(x) \neq 0 \\ \text{смјена} \quad y = y_1 + \frac{1}{z} \\ z = z(x) \end{gathered}}$$

$$y = y_1 + \frac{1}{z} = -\frac{1}{x} + \frac{1}{z}, \quad y' = \frac{1}{x^2} - \frac{z'}{z^2}$$

$$2x^2\left(\frac{1}{x^2} - \frac{z'}{z^2}\right) = x^2\left(-\frac{1}{x}+\frac{1}{z}\right)^2+1$$

$$\cancel{2} - \frac{2x^2z'}{z^2} = \cancel{1} - \frac{2x}{z} + \frac{x^2}{z^2} + \cancel{1} \quad /\cdot z^2$$

$$-2x^2z' = -2xz + x^2$$

$$-2x^2z' + 2xz = x^2 \quad \Big| : (-2x^2)$$

$$z' - \frac{1}{x} z = -\frac{1}{2} \qquad \text{лин. диф. јед.}$$

$$z(x) = e^{\int \frac{dx}{x}} \left(c - \frac{1}{2} \int e^{-\int \frac{dx}{x}} dx \right) = e^{\ln|x|} \left(c - \frac{1}{2} \int e^{-\ln|x|} dx \right)$$

$$= x \left(c - \frac{1}{2} \int \frac{dx}{x} \right) = x \left(c - \frac{1}{2} \ln|x| \right) = \frac{x(c - \ln|x|)}{2}$$

$$\boxed{y(x) = -\frac{1}{x} + \frac{2}{x(c - \ln|x|)}}$$

Зад: (8.09.2010.) Дата је диференцијална једначина

$(x^3-1)y' = 2xy^2 - x^2y - 1$. Одредити њен партикуларни интеграл у облику $y = ax+b$, а затим наћи општи интеграл.

Рјешење: $y' + \frac{x^2}{x^3-1}y = \frac{1}{1-x^3} + \frac{2x}{x^3-1}y^2$ - Рикатијеве диф. јед.

$y_p' = a$

$a(x^3-1) = 2x(ax+b)^2 - x^2(ax+b) - 1$

$ax^3 - a = 2x(a^2x^2 + 2abx + b^2) - ax^3 - bx^2 - 1$

$ax^3 - a = 2a^2x^3 + 4abx^2 + 2b^2x - ax^3 - bx^2 - 1$

$(2a - 2a^2)x^3 + (-4ab + b)x^2 - 2b^2x - a + 1 = 0$

$2a(1-a)x^3 + b(1-4a)x^2 - 2b^2x + 1 - a = 0$ $\Rightarrow$ $\boxed{a=1}$ $\boxed{b=0}$

$a=0 \vee a=1$, $b=0 \vee a=\frac{1}{4}$, $a=1$

$y_p = x$, смјена $y = y_p + \frac{1}{z} = x + \frac{1}{z}$

$y' = 1 - \frac{z'}{z^2}$

$(x^3-1)\left(1 - \frac{z'}{z^2}\right) = 2x\left(x + \frac{1}{z}\right)^2 - x^2\left(x + \frac{1}{z}\right) - 1$

$x^3 - \cancel{1} - (x^3-1)\frac{z'}{z^2} = 2x\left(x^2 + \frac{2x}{z} + \frac{1}{z^2}\right) - x^3 - \frac{x^2}{z} - \cancel{1}$

$\cancel{x^3} - (x^3-1)\frac{z'}{z^2} = \cancel{2x^3} + \frac{4x^2}{z} + \frac{2x}{z^2} \cancel{-x^3} - \frac{x^2}{z}$

$-(x^3-1)\frac{z'}{z^2} = \frac{3x^2}{z} + \frac{2x}{z^2} \quad / \cdot z^2$

$$z' + \frac{3x^2}{x^3-1} z = \frac{2x}{1-x^3}$$ 1. A. J.

$$z(x) = e^{-\int \frac{3x^2}{x^3-1}dx} \left(c + \int \frac{2x}{1-x^3} \cdot e^{\int \frac{3x^2}{x^3-1}dx} dx \right)$$

$$z(x) = e^{-\ln|x^3-1|} \left(c + \int \frac{2x}{1-x^3} e^{\ln|x^3-1|} dx \right)$$

$$z(x) = \frac{1}{x^3-1} \left(c + \int \frac{2x(x^3-1)}{1-x^3} dx \right) = \frac{1}{x^3-1} \left(c - \int 2x\,dx \right)$$

$$z(x) = \frac{1}{x^3-1} (c - x^2)$$

$$\boxed{y = x + \frac{x^3-1}{c-x^2} = \frac{cx-1}{c-x^2}}$$

ЗАД: (4.07.2014.)

Одредити опште рјешење диференцијалне једначине

$$\left(\frac{x}{\sin y}+2\right)dx+\frac{(x^2+1)\cos y}{\cos 2y-1}\,dy=0$$

Рјешење:

$$\frac{\partial P}{\partial y}=\frac{-x\cos y}{\sin^2 y}\quad,\quad \frac{\partial Q}{\partial x}=\frac{\cos y}{\cos 2y-1}\cdot 2x$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\Rightarrow \frac{-x\cos y}{\sin^2 y}=\frac{\cancel{2}x\cos y}{-\cancel{2}\sin^2 y}$$

$$u=\int P\,dx+\varphi(y)$$

$$u=\int\left(\frac{x}{\sin y}+2\right)dx+\varphi(y)=\frac{1}{\sin y}\cdot\frac{x^2}{2}+2x+\varphi(y)$$

$$\frac{\partial u}{\partial y}=Q\Rightarrow \frac{x^2}{2}\cdot\frac{-\cos y}{\sin^2 y}+\varphi'(y)=\frac{(x^2+1)\cos y}{\cos 2y-1}$$

$$-\frac{x^2}{2}\cdot\frac{\cos y}{\sin^2 y}+\varphi'(y)=\cancel{\frac{x^2\cos y}{-2\sin^2 y}}+\frac{\cos y}{-2\sin^2 y}\Rightarrow \varphi(y)=-\frac{1}{2}\int\frac{\cos y}{\sin^2 y}\,dy$$

$$\varphi(y)=-\frac{1}{2}\cdot\frac{(\sin y)^{-1}}{-1}+C=\frac{1}{2\sin y}+C$$

$$\frac{x^2}{2\sin y}+2x+\frac{1}{2\sin y}=C$$

$$\boxed{x^2+1=2(C-2x)\sin y}$$

ЗАД: (21.01.2015.) Одредити опште рјешење диференцијалне једначине $(2xy^2-y)dx+(y^2+x+y)dy=0.$

Рјешење: $P=2xy^2-y \quad , \quad Q=y^2+x+y$

$$\frac{\partial P}{\partial y}=4xy-1 , \quad \frac{\partial Q}{\partial x}=1 \quad \Rightarrow \quad \frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x}$$

$$\frac{\frac{d\lambda}{\lambda}}{dy}=\frac{\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial Q}}{P} \Rightarrow \frac{\frac{d\lambda}{\lambda}}{dy}=\frac{2-4xy}{2xy^2-y}=\frac{2\cancel{(1-2xy)}}{-y\cancel{(1-2xy)}}=-\frac{2}{y}$$

$$\frac{d\lambda}{\lambda}=-\frac{2}{y}dy \quad \Rightarrow \quad \ln|\lambda|=-2\ln|y| \quad \Rightarrow \quad \boxed{\lambda=y^{-2}}$$

$$P=y^{-2}(2xy^2-y)=2x-y^{-1}$$

$$Q=y^{-2}(y^2+x+y)=1+xy^{-2}+y^{-1}$$

$$\frac{\partial P}{\partial y}=y^{-2} , \quad \frac{\partial Q}{\partial x}=y^{-2}$$

$$u=\int(2x-y^{-1})dx+\varphi(y)=x^2-\frac{x}{y}+\varphi(y)$$

$$\frac{\partial u}{\partial y}=Q \quad \Rightarrow \quad \cancel{\frac{x}{y^2}}+\varphi'(y)=1+\cancel{xy^{-2}}+y^{-1} \quad \Rightarrow \quad \varphi'(y)=1+y^{-1}$$

$$\varphi(y)=\int\left(1+\frac{1}{y}\right)dy=y+\ln|y|+C$$

$$\boxed{x^2-\frac{x}{y}+y+\ln|y|=C}$$

ЗАД: (21.06.2013.) Показати да диференцијална једначина

$$(x-xy)dx + (x^2+y)dy = 0$$

има интеграциони фактор облика $\lambda = \lambda(x^2+y^2)$, па затим наћи њено опште рјешење.

Рјешење:

$$P = \lambda(x^2+y^2)(x-xy) \quad , \quad Q = \lambda(x^2+y^2)(x^2+y)$$

$$\frac{\partial P}{\partial y} = \lambda'(x^2+y^2)\cdot 2y(x-xy) + \lambda(x^2+y^2)(-x)$$

$$\frac{\partial Q}{\partial x} = \lambda'(x^2+y^2)\cdot 2x(x^2+y) + \lambda(x^2+y^2)\cdot 2x$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \lambda'(x^2+y^2)(\cancel{2xy} - 2xy^2 - 2x^3 - \cancel{2xy}) + \lambda(x^2+y^2)(-x-2x) = 0$$

$$\lambda'(x^2+y^2)(-2x)(x^2+y^2) = 3x\,\lambda(x^2+y^2)$$

$$\frac{d\lambda(x^2+y^2)}{d(x^2+y^2)}(-2\cancel{x})(x^2+y^2) = 3\cancel{x}\,\lambda(x^2+y^2)$$

$$\frac{d\lambda(x^2+y^2)}{\lambda(x^2+y^2)} = -\frac{3}{2}\,\frac{d(x^2+y^2)}{x^2+y^2} \quad \Big| \int$$

$$\ln|\lambda| = -\frac{3}{2}\ln(x^2+y^2) \quad \Rightarrow \quad \boxed{\lambda = (x^2+y^2)^{-3/2}}$$

$$P = (x^2+y^2)^{-3/2}(x-xy)$$

$$\frac{\partial P}{\partial y} = -\frac{3}{2}(x^2+y^2)^{-5/2}\cdot 2y(x-xy) + (x^2+y^2)^{-3/2}(-x)$$

$$= (x^2+y^2)^{-5/2}\left[-3y(x-xy) + (x^2+y^2)(-x)\right]$$

$$= (x^2+y^2)^{-5/2}(-3xy + 2xy^2 - x^3)$$

$$Q = (x^2+y^2)^{-3/2}(x^2+y)$$

$$\frac{\partial Q}{\partial x} = -\frac{3}{2}(x^2+y^2)^{-5/2}\cdot 2x\,(x^2+y) + (x^2+y^2)^{-3/2}\cdot 2x$$

$$= (x^2+y^2)^{-5/2}(-x^3-3xy+2xy^2)$$

$$u = \int P(x,y)\,dx + \varphi(y) = \int \frac{x-xy}{(x^2+y^2)^{3/2}}\,dx + \varphi(y) = \int \frac{x\,dx}{(x^2+y^2)^{3/2}} - y\int\frac{x\,dx}{(x^2+y^2)^{3/2}}$$

$$+\varphi(y) = \left| \begin{matrix} x^2+y^2=t \\ 2x\,dx = dt \end{matrix} \right| = \dots = \underline{(x^2+y^2)^{-1/2}(y-1) + \varphi(y)}$$

$$\frac{\partial u}{\partial y} = Q$$

$$-\frac{1}{\cancel{2}}(x^2+y^2)^{-3/2}\cdot \cancel{2}y(y-1) + (x^2+y^2)^{-1/2}\cdot 1 + \varphi'(y) = (x^2+y^2)^{-3/2}(x^2+y)$$

$$\cancel{(x^2+y^2)^{-3/2}}(\cancel{-y^2}+y+x^2+\cancel{y^2}) + \varphi'(y) = \cancel{(x^2+y^2)^{-3/2}}(x^2+y)$$

$$\varphi'(y) = 0 \Rightarrow \varphi(y) = C$$

$$\boxed{(x^2+y^2)^{-1/2}(y-1) = C}$$ општe рјешење

Зад: (6.02.2015.)

Одредити константу $a \in \mathbb{R}$ тако да диференцијална једначина $x^2y'' - x(x-1)y' - y = 0$ има партикуларно рјешење облика $y_1 = \frac{e^{ax}}{x}$, па затим одредити њено опште рјешење.

Рјешење: $y_1' = \frac{ae^{ax}\cdot x - e^{ax}}{x^2} = \frac{e^{ax}(ax-1)}{x^2}$

$$y_1'' = \frac{(a^2e^{ax}\cdot x + \cancel{ae^{ax}} - \cancel{ae^{ax}})x^{\cancel{2}} - (axe^{ax} - e^{ax})\cdot 2\cancel{x}}{x^{\cancel{4}}}$$

$$y_1'' = \frac{a^2x^2e^{ax} - 2axe^{ax} + 2e^{ax}}{x^3} = \boxed{\frac{e^{ax}(a^2x^2 - 2ax + 2)}{x^3}}$$

$$e^{ax}\frac{(a^2x^2 - 2ax + 2)}{x} - (x^{\cancel{2}} - \cancel{x})e^{ax}\frac{(ax-1)}{x^{\cancel{2}}} - \frac{e^{ax}}{x} = 0 \quad / \cdot \frac{x}{e^{ax}}$$

$$a^2x^2 - 2ax + 2 - (x-1)(ax-1) - 1 = 0$$

$$a^2x^2 - 2ax + 2 - ax^2 + x + ax - \cancel{1} - \cancel{1} = 0$$

$$a^2x^2 - 2ax + \cancel{2} - ax^2$$

$$(a^2 - a)x^2 + x(1-a) = 0$$

$$a(a-1)x^2 + x(1-a) = 0 \quad \Rightarrow \quad \boxed{a = 1}$$

$$y'' + f_1(x)y' + f_2(x)y = 0$$

, LIOUVILLOVA FORMULA

$$y_2(x) = y_1(x)\int \frac{1}{y_1^2(x)} e^{-\int f_1(x)dx} dx$$

$$y_{o.p} = C_1y_1 + C_2y_2$$

$$y'' - \frac{x-1}{x}y' - \frac{1}{x^2}y = 0 \quad , \quad f_1(x) = \frac{1-x}{x}$$

$$\boxed{y_1(x) = \frac{e^x}{x}}$$

$$y_2(x) = \frac{e^x}{x}\int \frac{x^2}{e^{2x}} \cdot e^{\int \frac{x-1}{x}dx} = \frac{e^x}{x}\int \frac{x^2}{e^{2x}} \cdot e^{\int (1-\frac{1}{x})dx} dx$$

$$= \frac{e^x}{x}\int \frac{x^2}{e^{2x}} \cdot e^{x - \ln|x|} dx = \frac{e^x}{x}\int \frac{\cancel{x^2}}{e^{2x}} \cdot \cancel{e^x} \cdot \frac{1}{\cancel{x}} dx = \frac{e^x}{x}\int \frac{x\,dx}{e^x}$$

$$y_2(x) = \frac{e^x}{x}\int x e^{-x} dx \quad \left| \begin{array}{l} u = x \quad , du = dx \\ dv = e^{-x} dx \, , \quad \underline{v = -e^{-x}} \end{array} \right|$$

$$= \frac{e^x}{x}\left(-x e^{-x} + \int e^{-x} dx\right) = \frac{e^x}{x}\left(-x e^{-x} - e^{-x}\right) = \frac{-x-1}{x}$$

$$\boxed{y_{o.p} = c_1 \frac{e^x}{x} + c_2 \frac{x+1}{x}}$$

ЗАД: (15.04.2015.)

Одредити опште рјешење диференцијалне једначине

$$x(1-x)y'' + (2x^2-1)y' + 2(1-2x)y = 0$$

ако је познато да она има партикуларно рјешење у облику полинома другог степена.

Рјешење: $y_p = ax^2+bx+c$, $y_p' = 2ax+b$, $y_p'' = 2a$

$$(x-x^2)\cdot 2a + (2x^2-1)(2ax+b) + 2(1-2x)(ax^2+bx+c) = 0$$

$$\cancel{2ax} - \cancel{2ax^2} + \cancel{4ax^3} + 2bx^2 - \cancel{2ax} - b + \cancel{2ax^2} + 2bx + 2c - \cancel{4ax^3} - 4bx^2 - 4cx = 0$$

$$-2bx^2 + 2bx - 4cx - b + 2c = 0$$

$$\begin{aligned} b &= 0 \\ 2b-4c &= 0 \\ -b+2c &= 0 \end{aligned} \Rightarrow \begin{cases} b = 0 \\ c = 0 \\ a - \text{произвољно} \end{cases}, \quad \text{нека је } a = 1$$

$$y_p = x^2$$

$$y = y_p \cdot z = x^2 z \quad , \quad y' = 2xz + x^2 z' \quad , \quad y'' = x^2 z'' + 4xz' + 2z$$

$$(x-x^2)(x^2z''+4xz'+2z) + (2x^2-1)(2xz+x^2z') + (2-4x)\cdot x^2 z = 0$$

$$z''(x^3-x^4) + z'(4x^2-4x^3+2x^4-x^2) + z(\cancel{2x}-\cancel{2x^2}+\cancel{4x^3}-\cancel{2x}+\cancel{2x^2}-\cancel{4x^3}) = 0$$

$$\frac{z''}{z'} = -\frac{2x^4-4x^3+3}{x^2-x} \Rightarrow \frac{z''}{z'} = \frac{2x^2-4x+3}{x^2-x} = 2 + \frac{-2x+3}{x^2-x} \quad \Big/ \int$$

$$\ln|z'| = 2x - \int \frac{2x-3}{x^2-x}dx = 2x - 3\int\frac{dx}{x} + \int\frac{dx}{x-1}$$

$$\ln|z'| = 2x - 3\ln|x| + \ln|x-1| + \ln c_1 = \ln e^{2x}\cdot c_1 \frac{(x-1)}{x^3}$$

$$z'(x) = \frac{c_1 e^{2x}(x-1)}{x^3}$$

$$z(x) = c_1 \int e^{2x}\left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx = c_1\left[\int \frac{e^{2x}dx}{x^2} - \int \frac{e^{2x}}{x^3}dx\right]$$

$$\left| \begin{matrix} u = \cancel{e^{2x}}, du = 2e^{2x} dx \\ v = -\frac{1}{2x^2} \end{matrix} \right| = c_1 \left[\cancel{\int \frac{e^{2x} dx}{x^2}} + \frac{e^{2x}}{2x^2} - \cancel{\int \frac{e^{2x}}{x^2} dx} + c_2 \right]$$

$$= c_1 \left(\frac{e^{2x}}{2x^2} + c_2 \right)$$

$$y = x^2 c_1 \left(\frac{e^{2x}}{2x^2} + c_2 \right) = \frac{c_1}{2} e^{2x} + c_1 c_2 x^2 \Rightarrow \boxed{y = c_1 e^{2x} + c_2 x^2}$$

ЗАД: (19.10.2014.)

Наћи опште рјешење диференцијалне једначине

$$y'' - y' - 2y = e^{2x}\cos^2 x$$

Рјешење:

$$y'' - y' - 2y = e^{2x}\cdot\frac{1+\cos 2x}{2} = \frac{e^{2x}}{2} + \frac{1}{2}e^{2x}\cdot\cos 2x$$

$$\lambda^2 - \lambda - 2 = 0 \qquad \lambda_1 = 2, \quad \lambda_2 = -1$$

$$(\lambda-2)(\lambda+1) = 0$$

$$y_H = C_1 e^{2x} + C_2 e^{-x}$$

МЕТОДА ВАРИЈАЦИЈЕ КОНСТАНТИ

$$y_H = C_1(x)e^{2x} + C_2(x)e^{-x}$$

$$\left.\begin{aligned} C_1'(x)e^{2x} + C_2'(x)e^{-x} &= 0 \\ 2C_1'(x)e^{2x} - C_2'(x)e^{-x} &= e^{2x}\cos^2 x \end{aligned}\right\} + \Rightarrow$$

$$3C_1'(x)\cancel{e^{2x}} = \cancel{e^{2x}}\cdot\cos^2 x \Rightarrow C_1'(x) = \frac{1}{3}\cos^2 x \Rightarrow$$

$$C_1(x) = \frac{1}{3}\int\cos^2 x\,dx = \frac{1}{3}\int\frac{1+\cos 2x}{2}dx = \frac{1}{6}\left(\int dx + \int\cos 2x\,dx\right)$$

$$= \frac{1}{6}\left(x + \frac{1}{2}\sin 2x\right) + C_1 = \frac{1}{6}x + \frac{1}{12}\sin 2x + C_1$$

$$C_1(x) = \frac{1}{6}x + \frac{1}{12}\sin 2x + C_1$$

$$C_2'(x)e^{-x} = -C_1'(x)e^{2x} \Rightarrow C_2'(x) = -\frac{1}{3}\cos^2 x\cdot e^{3x}$$

$$C_2(x) = -\frac{1}{3}\int\frac{1+\cos 2x}{2}e^{3x} = -\frac{1}{6}\left[\int e^{3x}dx + \int\cos 2x\cdot e^{3x}dx\right]$$

$$C_2(x) = -\frac{1}{6}\left[\frac{1}{3}e^{3x} + I_1\right]$$

$$I_1 = \int \cos 2x \cdot e^{3x}\,dx = \left| \begin{array}{ll} u = \cos 2x \Rightarrow & du = -2\sin 2x\,dx \\ dv = e^{3x}dx \Rightarrow & v = \frac{1}{3}e^{3x} \end{array} \right|$$

$$= \frac{\cos 2x}{3}e^{3x} - \frac{2}{3}\int \sin 2x \cdot e^{3x}\,dx \quad \left| \begin{array}{ll} u = \sin 2x, & du = 2\cos 2x \\ v = \frac{1}{3}e^{3x} & \end{array} \right|$$

$$= \frac{\cos 2x}{3}e^{3x} - \frac{2}{3}\left(\frac{\sin 2x}{3}e^{3x} - \frac{2}{3}\int \cos 2x \cdot e^{3x}\,dx\right)$$

$$I_1 = \frac{\cos 2x}{3}e^{3x} - \frac{2}{9}\sin 2x \cdot e^{3x} + \frac{4}{9}I_1$$

$$\frac{5}{9}I_1 = \frac{\cos 2x}{3}e^{3x} - \frac{2}{9}\sin 2x \cdot e^{3x}$$

$$I_1 = \frac{9}{5}\left(\frac{\cos 2x}{3} - \frac{2}{9}\sin 2x\right)\cdot e^{3x} = \frac{e^{3x}}{5}\left(3\cos 2x - 2\sin 2x\right)$$

$$C_2(x) = -\frac{1}{6}\left[\frac{1}{3}e^{3x} + \frac{e^{3x}}{5}\left(3\cos 2x - 2\sin 2x\right)\right] + C_2$$

$$\boxed{C_2(x) = -\frac{e^{3x}}{6}\left[\frac{1}{3} + \frac{1}{5}\left(3\cos 2x - 2\sin 2x\right)\right] + C_2}$$

$$\boxed{y_{o.p.} = e^{2x}\left(\frac{1}{6}x - \frac{1}{18} + \frac{3}{20}\sin 2x - \frac{1}{10}\cos 2x\right) + C_1e^{2x} + C_2e^{-x}}$$

Зад: (9.09.2014.)
Наћи опште рјешење диференцијалне једначине

$y'' + y' = x - \sin 2x$
па затим одредити партикуларно рјешење које задовољава услове $y(0) = 2$, $y'(0) = 1$.

Рјешење: $\lambda^2 + \lambda = 0 \Rightarrow \lambda(\lambda+1) \Rightarrow \boxed{\lambda_1 = 0, \quad \lambda_2 = -1}$

$y_H = C_1 + C_2 e^{-x}$

$y_{o.p} = y_H + y_{p_1} + y_{p_2}$

$y_{p_1} = (Ax + B)x = Ax^2 + Bx$

$y_{p_1}' = 2Ax + B, \quad y_{p_1}'' = 2A$

$2A + 2Ax + B = x \Rightarrow \boxed{A = \frac{1}{2}}$ $\quad 2A + B = 0$ $\quad \boxed{B = -1}$

$\boxed{y_{p_1} = \frac{1}{2}x^2 - x}$

$y_{p_2} = C \cdot \cos 2x + D \sin 2x$

$y_{p_2}' = -2C \sin 2x + 2D \cos 2x, \quad y_{p_2}'' = -4C \cos 2x - 4D \sin 2x$

$-4C \cos 2x - 4D \sin 2x - 2C \sin 2x + 2D \cos 2x = -\sin 2x$

$-4C + 2D = 0 \quad / \cdot 2$
$-4D - 2C = -1$

$-10C = -1 \Rightarrow \boxed{C = \frac{1}{10}}$ $\quad \boxed{D = \frac{1}{5}}$

$\boxed{y_{p_2} = \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x}$

$\boxed{y_{o.p.} = C_1 + C_2 e^{-x} + \frac{1}{2}x^2 - x + \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x}$

ЗАД: (10.09.2013.)

Одредити опште рјешење диференцијалне једначине

$$(2+x)^2 y'' - 3(2+x)y' + 4y = (2+x)^2$$

Рјешење:

Ојлерова диференцијална једначина

$$(ax+b)^n y^{(n)} + A_1(ax+b)^{n-1} y^{(n-1)} + \dots + A_{n-1}(ax+b)y' + A_n y = f(x)$$

смјена: $ax+b = e^t$

$$y' = a e^{-t} y'_t \quad , \quad y'' = a^2 e^{-2t}(y''_t - y'_t) \dots$$

$$|2+x| = e^t \Rightarrow t = \ln|2+x|$$

$$y'_x = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = y'_t \cdot e^{-t}$$

$$y''_x = \frac{d}{dt}(y'_x)\cdot\frac{dt}{dx} = (y''_t \cdot e^{-t} - y'_t \cdot e^{-t})\cdot e^{-t} = (y''_t - y'_t)e^{-2t}$$

$$e^{2t}\cdot(y''_t - y'_t)e^{-2t} - 3\cdot e^{t}\cdot y'_t \cdot e^{-t} + 4y = e^{2t}$$

$$y''_t - 4y'_t + 4y = e^{2t}$$

карактеристична једначина $\lambda^2 - 4\lambda + 4 = 0$

$$(\lambda-2)^2 = 0 \Rightarrow \lambda_{1,2} = 2$$

$$y_H = C_1 e^{2t} + C_2 t e^{2t}$$

$$y_p = At^2 e^{2t} \Rightarrow y_p' = 2At e^{2t} + At^2 \cdot 2e^{2t}$$

$$y_p' = e^{2t}(2At + 2At^2)$$

$$y_p'' = 2e^{2t}(2At + 2At^2) + e^{2t}(2A + 4At) = e^{2t}(4At^2 + 8At + 2A)$$

$$\cancel{4At^2} + \cancel{8At} + 2A - \cancel{8At} - \cancel{8At^2} + \cancel{4At^2} = 1 \Rightarrow \boxed{A = \frac{1}{2}}$$

$$y_{op} = C_1 e^{2t} + C_2 t e^{2t} + \frac{t^2}{2} e^{2t} = e^{2t}\left(C_1 + C_2 t + \frac{t^2}{2}\right)$$

$$y_{op.} = (2+x)^2\left(C_1 + C_2 \cdot \ln|2+x| + \frac{1}{2}\ln^2|2+x|\right)$$