ОДРЕЂЕНИ ИНТЕГРАЛИ

ЗАД1: ИЗРАЧУНАТИ $\int_0^1 x(2-x^2)^{12}\,dx$

РЈЕШЕЊЕ: ПОДИНТЕГРАЛНА ФУНКЦИЈА НЕПРЕКИДНА, ПА КОРИСТИМО ЊУТН-ЛАЈБНИЦОВУ ФОРМУЛУ

смјена: $\left|\begin{array}{l} 2-x^2=t \\ -2x\,dx=dt \end{array}\right| = -\frac{1}{2}\int t^{12}\,dt = -\frac{1}{2}\cdot\frac{t^{13}}{13} = -\frac{1}{26}(2-x^2)^{13}\Big|_0^1$

$$= -\frac{1}{26}\left[(2-1)^{13}-2^{13}\right] = -\frac{1}{26}(1-2^{13}) = -\frac{1}{26}(1-8192) = \boxed{\frac{8191}{26}}$$

ЗАД2: ИЗРАЧУНАТИ $\int_{-1}^1 \frac{x\,dx}{x^2+x+1} = \frac{1}{2}\int_{-1}^1 \frac{2x\,dx}{x^2+x+1} = \frac{1}{2}\left[\int_{-1}^1 \frac{2x+1}{x^2+x+1}\,dx - \int \frac{dx}{x^2+x+1}\right]$

$$= \frac{1}{2}\left[\ln(x^2+x+1)\Big|_{-1}^1 - \int_{-1}^1 \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}\right]$$

$$= \frac{1}{2}\left[\ln 3 - \ln 1 - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}\Big|_{-1}^1\right] = \frac{1}{2}\ln 3 - \frac{\sqrt{3}}{3}\left(\operatorname{arctg}\sqrt{3} + \operatorname{arctg}\frac{\sqrt{3}}{3}\right)$$

$$= \frac{1}{2}\ln 3 - \frac{\sqrt{3}}{3}\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \boxed{\frac{1}{2}\ln 3 - \frac{\sqrt{3}\pi}{6}}$$

ЗАД3: ИЗРАЧУНАТИ $\int_1^2 \frac{dx}{1+\sqrt{2x-x^2}}$

РЈЕШЕЊЕ: $\int_1^2 \frac{dx}{1+\sqrt{1-(1-x)^2}}$ $\left|\begin{array}{l} 1-x=\sin t \\ -dx=\cos t\,dt \\ \begin{array}{c|c|c} x & 1 & 2 \\ \hline t & 0 & -\frac{\pi}{2} \end{array} \end{array}\right| = -\int_0^{-\frac{\pi}{2}} \frac{\cos t\,dt}{1+\cos t}$

$$= \int_{-\frac{\pi}{2}}^0 \frac{\cos t+1-1}{1+\cos t}\,dt = \int_{-\frac{\pi}{2}}^0 dt - \int_{-\frac{\pi}{2}}^0 \frac{dt}{2\cos^2\frac{t}{2}} = \frac{\pi}{2} - \frac{1}{2}\left(\operatorname{tg}\frac{t}{2}\right)\Big|_{-\frac{\pi}{2}}^0$$

$$= \frac{\pi}{2} - \frac{1}{2}\left(\operatorname{tg}0 - \operatorname{tg}\left(-\frac{\pi}{4}\right)\right) = \boxed{\frac{\pi}{2}-1}$$

ЗАДАЧ: ИЗРАЧУНАТИ: $\displaystyle\int_0^{2\pi} \frac{dx}{2\sin x - \cos x + 5}$

РЕШЕЊЕ:

смене: $tg\frac{x}{2} = t$, $tg\frac{x}{2}$ није дефинисан у π

па је $I = \int_0^{\pi-} + \int_{\pi+}^{2\pi} = F(\pi-) - F(0) + F(2\pi) - F(\pi+)$

$\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$

$$\int \frac{\frac{2dt}{1+t^2}}{\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} + 5} = \int \frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{4t-1+t^2+5+5t^2}{\cancel{1+t^2}}} = \int \frac{2dt}{6t^2+4t+4}$$

$$= \int \frac{dt}{3t^2+2t+2} = \frac{1}{3}\int \frac{dt}{(t+\frac{1}{3})^2 + \frac{5}{9}} = \frac{1}{\sqrt5}\, arctg\, \frac{3t+1}{\sqrt5}$$

$$I_1 = \frac{1}{\sqrt5}\, arctg\, \frac{3tg\frac{x}{2}+1}{\sqrt5}\Big|_0^{\pi} = \frac{1}{\sqrt5}\, arctg\, \infty - \frac{1}{\sqrt5}\, arctg\, \frac{1}{\sqrt5} = \frac{\sqrt5}{5}\left(\frac{\pi}{2} - arctg\frac{1}{\sqrt5}\right)$$

$$I_2 = \frac{\sqrt5}{5}\left(arctg\, \frac{3tg\pi+1}{\sqrt5} - arctg\, \frac{\cancel{3tg\frac{\pi}{2}+1}}{\sqrt5}\right) = \frac{\sqrt5}{5}\left(arctg\frac{1}{\sqrt5} + \frac{\pi}{2}\right)$$

$-\infty$

$$I = \frac{1}{\sqrt5}\cdot\frac{\pi}{2} - \cancel{\frac{1}{\sqrt5}\, arctg\frac{1}{\sqrt5}} + \cancel{\frac{1}{\sqrt5}\, arctg\frac{1}{\sqrt5}} + \frac{1}{\sqrt5}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt5}$$

P: $\boxed{\frac{\pi}{\sqrt5}}$

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ЗАД 5: ИЗРАЧУНАТИ $\displaystyle\int_0^{2\pi}\frac{dx}{\sin^4x+\cos^4x}=\int_0^{2\pi}\frac{dx}{(\sin^2x+\cos^2x)^2-2\sin^2x\cos^2x}$

$$=\int_0^{2\pi}\frac{dx}{1-2\sin^2x\cos^2x}=\int_0^{2\pi}\frac{dx}{1-\frac{1}{2}\sin^2 2x}=\int_0^{2\pi}\frac{dx}{\frac{2-\sin^2 2x}{2}}$$

$$=\int_0^{2\pi}\frac{dx}{\frac{1+1-\sin^2 2x}{2}}=\int_0^{2\pi}\frac{dx}{\frac{1+\cos^2 2x}{2}}=2\int_0^{2\pi}\frac{dx}{1+\cos^2 2x}=\left|\begin{array}{l} \operatorname{tg}2x=t\\ dx=\frac{dt}{2(1+t^2)}\\ \cos^2 2x=\frac{1}{1+t^2}\end{array}\right.$$

$$x\in\left(-\frac{\pi}{4}+\frac{k\pi}{2},\ \frac{\pi}{4}+\frac{k\pi}{2}\right)$$

$k=0,\quad x\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

$k=1\quad x\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$

$k=2\quad x\in\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)$

$k=3\quad x\in\left(\frac{5\pi}{4},\frac{7\pi}{4}\right)$

$k=4\quad x\in\left(\frac{7\pi}{4},2\pi\right)$

$$\boxed{=\int\frac{dt}{t^2+2}=\frac{\sqrt2}{2}\operatorname{arctg}\frac{\operatorname{tg}2x}{\sqrt2}}$$

$$\frac{1}{\cos^2 2x}\cdot 2dx=dt$$

$$2x=\operatorname{arctg}t$$

$$dx=\frac{dt}{2(1+t^2)}$$

$$I=\frac{\sqrt2}{2}\left[\int_0^{\frac{\pi}{4}}+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}+\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}+\int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}}+\int_{\frac{7\pi}{4}}^{2\pi}\right]$$

$$=\frac{\sqrt2}{2}\left[\operatorname{arctg}\frac{\operatorname{tg}\frac{\pi}{2}}{\sqrt2}^{+\infty}\right.$$

$$-\frac{\pi}{2}+k\pi<2x<\frac{\pi}{2}+k\pi\ /:2$$

$$-\frac{\pi}{4}+\frac{k\pi}{2}<x<\frac{\pi}{4}+\frac{k\pi}{2}$$

$$\boxed{Р:\ 2\pi\sqrt2}$$

Зад6: Израчунати $\displaystyle\int_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,dx$

Решење: $u = \arcsin\sqrt{\frac{x}{1+x}}$, $\quad dv = dx \Rightarrow \boxed{v = x}$

$$du = \frac{1}{\sqrt{1-\frac{x}{1+x}}} \cdot \frac{1}{2}\left(\frac{x}{1+x}\right)^{-1/2} \cdot \frac{1+x-x}{(1+x)^2}\,dx$$

$$du = \sqrt{1+x} \cdot \frac{1}{2}\sqrt{\frac{1+x}{x}} \cdot \frac{dx}{(1+x)^2}$$

(јер $\sqrt{(1+x)^2} = 1+x$, $1+x>0$ на $[0,3]$)

$$du = \frac{\cancel{1+x}}{2\sqrt{x}} \cdot \frac{dx}{(1+x)^{\cancel{2}}}, \quad \boxed{du = \frac{dx}{2(1+x)\sqrt{x}}}$$

$$I = x\cdot\arcsin\sqrt{\frac{x}{1+x}}\,\Big|_0^3 - \int_0^3 \frac{x\,dx}{2(1+x)\sqrt{x}}$$

$$I = 3\cdot\arcsin\sqrt{\frac{3}{4}} - \underbrace{\int_0^3 \frac{x\,dx}{2(1+x)\sqrt{x}}}_{I_1}$$

$$I_1 = \left| \begin{array}{c} x = t^2 \\ dx = 2t\,dt \\ \begin{array}{c|c|c} x & 0 & 3 \\ t & 0 & \sqrt{3} \end{array} \end{array} \right| = \int_0^{\sqrt{3}} \frac{t^2\cdot \cancel{2t}\,dt}{\cancel{2}(1+t^2)\,\cancel{t}} = \int_0^{\sqrt{3}} \frac{t^2+1-1}{1+t^2}\,dt$$

$$= \int_0^{\sqrt{3}} dt - \int_0^{\sqrt{3}} \frac{dt}{1+t^2} = t\Big|_0^{\sqrt{3}} - \operatorname{arctg} t\Big|_0^{\sqrt{3}}$$

$$= \sqrt{3} - \operatorname{arctg}\sqrt{3} = \sqrt{3} - \frac{\pi}{3}$$

$$I = 3\cdot\arcsin\frac{\sqrt{3}}{2} - \sqrt{3} + \frac{\pi}{3} = 3\cdot\frac{\pi}{3} - \sqrt{3} + \frac{\pi}{3} = \boxed{\frac{4\pi}{3} - \sqrt{3}}$$

A3.

ЗАД 7: ИЗРАЧУНАТИ $\int_0^2 x\sqrt{x^2+4}\,\operatorname{arctg}\frac{x}{2}\,dx$

РЕШЕЊЕ: $u=\operatorname{arctg}\frac{x}{2} \Rightarrow du=\frac{1}{1+\frac{x^2}{4}}\cdot\frac{dx}{2}$

$$du=\frac{2\,dx}{x^2+4}$$

$$v=\int x\sqrt{x^2+4}\,dx=\left|\begin{matrix} x^2+4=t^2\\ 2x\,dx=2t\,dt\\ x\,dx=t\,dt\end{matrix}\right|=\int t^2\,dt=\frac{t^3}{3}=\frac{1}{3}(x^2+4)^{3/2}$$

$$I=\frac{1}{3}(x^2+4)^{3/2}\cdot\operatorname{arctg}\frac{x}{2}\Big|_0^2-\frac{2}{3}\int\frac{dx}{(x^2+4)(x^2+4)^{-3/2}}$$

$$I=\frac{1}{3}\left(8^{3/2}\operatorname{arctg}1\right)-\frac{2}{3}\int_0^2\sqrt{x^2+4}\,dx=\frac{1}{3}\,2^4\sqrt{2}\cdot\frac{\pi}{4}-\frac{2}{3}\left[x\sqrt{x^2+4}+4\ln\left(x+\sqrt{x^2+4}\right)\right]_0^2=\frac{4\sqrt{2}}{3}\pi-\frac{2}{3}\left[2\sqrt{8}+4\ln(2+\sqrt{8})-4\ln\sqrt{4}\right]$$

$$=\frac{4\sqrt{2}}{3}\pi-\frac{2}{3}\left[4\sqrt{2}+4\ln(2+2\sqrt{2})-4\ln 2\right]$$

$$=\frac{4\sqrt{2}}{3}\pi-\frac{2}{3}\left[4\sqrt{2}+4\cancel{\ln 2}+4\ln(1+\sqrt{2})\cancel{-4\ln 2}\right]$$

$$=\frac{4\sqrt{2}}{3}\pi-\frac{8}{3}\sqrt{2}-\frac{8}{3}\ln(1+\sqrt{2})=-\cancel{8}\sqrt{2}$$

ЗАД 8: ИЗРАЧУНАТИ: $\int_0^{\pi} e^{-\lambda x} \sin x \, dx$

РЈЕШЕЊЕ:

$$\left| \begin{array}{ll} u = e^{-\lambda x} & dv = \sin x \, dx \\ du = -\lambda e^{-\lambda x} dx & v = -\cos x \end{array} \right|$$

$$= \left[-\cos x \cdot e^{-\lambda x}\right]_0^{\pi} - \lambda \int_0^{\pi} e^{-\lambda x} \cos x \, dx = e^{-\lambda \pi} + 1 - \lambda \int_0^{\pi} e^{-\lambda x} \cos x \, dx$$

$$\left| \begin{array}{l} u = e^{-\lambda x} \\ dv = \cos x \, dx \end{array} \right| = e^{-\lambda\pi} + 1 - \lambda\left[\left(e^{-\lambda x} \sin x\right)_0^{\pi} + \lambda \int_0^{\pi} e^{-\lambda x} \sin x \, dx\right]$$

$$= e^{-\lambda \pi} + 1 - \lambda^2 \underbrace{\int_0^{\pi} e^{-\lambda x} \sin x \, dx}_{I}$$

$$\int_0^{\pi} e^{-\lambda x} \sin x \, dx = \frac{e^{-\lambda\pi} + 1}{\lambda^2 + 1}$$

ЗАД 9: ИЗРАЧУНАТИ $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

користимо $\boxed{\int_0^{\pi} x \, f(\sin x) \, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) \, dx}$

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx = \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \\ \begin{array}{c|c|c} x & 0 & \pi \\ \hline t & 1 & -1 \end{array} \end{array} \right|$$

$$= -\frac{\pi}{2} \int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2} \int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2} \operatorname{arctg} \cancel{(\cos x)} \Big|_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\operatorname{arctg} 1 - \operatorname{arctg}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$$

ЗАД 10: ИЗРАЧУНАТИ ВРИЈЕДНОСТ ИНТЕГРАЛА

$$I_n = \int_0^1 (1-x^2)^n \, dx \quad , \quad n = 1, 2, \dots$$

РЈЕШЕЊЕ:

$$I_n = \int_0^1 (1-x^2)^n dx = \left| \begin{array}{ll} u = (1-x^2)^n & dv = dx \\ du = -2n(1-x^2)^{n-1} x\,dx \ , & v = x \end{array} \right|$$

$$= \underbrace{\left[x(1-x^2)^n \right]_0^1}_{=0} + 2n \int_0^1 x^2 (1-x^2)^{n-1} dx = 2n \int_0^1 (x^2+1-1)(1-x^2)^{n-1} dx$$

$$= 2n \int_0^1 (x^2-1)(1-x^2)^{n-1} dx + 2n \int_0^1 (1-x^2)^{n-1} dx$$

$$= -2n \underbrace{\int_0^1 (1-x^2)^n dx}_{I_n} + 2n \underbrace{\int_0^1 (1-x^2)^{n-1} dx}_{I_{n-1}}$$

$$I_n = -2n I_n + 2n I_{n-1}$$

$$I_n = \frac{2n}{2n+1} I_{n-1} \quad , \quad n = 1, 2, \dots$$

$$I_1 = \int_0^1 (1-x^2) dx = \left[x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}$$

$$I_n = \frac{2n}{2n+1} I_{n-1} = \frac{2n}{2n+1} \cdot \frac{2(n-1)}{2(n-1)+1} I_{n-2}$$

$$I_n = \frac{2n}{2n+1} \cdot \frac{2(n-1)}{2(n-1)+1} \cdot \frac{2(n-2)}{2(n-2)+1} \cdots , \frac{2\cdot 2}{\underbrace{2\cdot 2+1}_{5}} \cdot \underbrace{I_1}_{=\frac{2}{3}}$$

$$I_n = \frac{2^n \cdot n!}{(2n+1)!!} = \frac{2^n \cdot n! \ 2\cdot 4 \cdot \ldots \cdot 2n}{(2n+1)!} = \frac{2^{2n} (n!)^2}{(2n+1)!}$$

ЗАД.: ИЗРАЧУНАТИ $I = \int_0^{200\pi} \sqrt{1-\cos 2x}\, dx$

РЈЕШЕЊЕ:

$I = \int_0^{200\pi} \sqrt{2\sin^2 x}\, dx = \sqrt{2} \int_0^{200\pi} |\sin x|\, dx$ [функција $|\sin x|$ је π - периодична]

$I = 200\sqrt{2} \int_0^{\pi} \sin x\, dx = 200\sqrt{2}\,(-\cos x)\Big|_0^{\pi} = 400\sqrt{2}$

ЗАД.: (И. 3.)

ИЗРАЧУНАТИ $I = \int_0^{10\pi} \sqrt{1+\sin 2x}\, dx$

РЈЕШЕЊЕ:

функција $1+\sin 2x$ - је π периодична

$I = 10\int_0^{\pi} \sqrt{(\sin x+\cos x)^2}\, dx = 10\int_0^{\pi} |\sin x+\cos x|\, dx$

$= 10\left(\int_0^{\frac{3\pi}{4}} (\sin x+\cos x)\, dx + \int_{\frac{3\pi}{4}}^{\pi} (-\sin x-\cos x)\, dx \right.$

$= 10\left(-\cos x\Big|_0^{\frac{3\pi}{4}} + \sin x\Big|_0^{\frac{3\pi}{4}} + \cos x\Big|_{\frac{3\pi}{4}}^{\pi} - \sin x\Big|_{\frac{3\pi}{4}}^{\pi}\right)$

$= 10\left(+\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2} - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = \underline{\underline{20\sqrt{2}}}$

НЕСВОЈСТВЕНИ ИНТЕГРАЛИ

ЗАД 1: ИЗРАЧУНАТИ $\displaystyle\int_2^{+\infty}\frac{dx}{x^2-2}$

РЈЕШЕЊЕ:

$$\int_2^{+\infty}\frac{dx}{x^2-2} = \lim_{a\to+\infty}\int_2^{a}\frac{dx}{x^2-2} = \frac{1}{2\sqrt2}\lim_{a\to+\infty}\left(\ln\frac{x-\sqrt2}{x+\sqrt2}\right)\Bigg|_2^a$$

$$= \frac{1}{2\sqrt2}\left[\lim_{a\to+\infty}\underbrace{\ln\underbrace{\frac{a-\sqrt2}{a+\sqrt2}}_{1}}_{0} - \ln\frac{2-\sqrt2}{2+\sqrt2}\right] = \underbrace{\frac{1}{2\sqrt2}\ln\frac{2+\sqrt2}{2-\sqrt2}}$$

ЗАД 2: ИЗРАЧУНАТИ $\displaystyle\int_0^{\pi}\frac{dx}{1+\sin^2x}$

РЈЕШЕЊЕ:

$$\int_0^{\pi}\frac{dx}{\cos^2x+2\sin^2x} = \int_0^{\frac{\pi}{2}-}\frac{dx}{\cos^2x\,(1+2tg^2x)} + \int_{\frac{\pi}{2}+}^{\pi}\frac{dx}{\cos^2x\,(1+2tg^2x)}$$

$$\left|\begin{array}{l} tg x = t \\ \dfrac{dx}{\cos^2x} = dt \end{array}\right.$$

x	0	$\frac{\pi}{2}-$	$\frac{\pi}{2}+$	π
t	0	$+\infty$	$-\infty$	0

$$= \int_0^{+\infty}\frac{dt}{1+2t^2} + \int_{-\infty}^{0}\frac{dt}{1+2t^2} = \int_0^{+\infty}\frac{dt}{1+(\sqrt2 t)^2} + \int_{-\infty}^{0}\frac{dt}{1+(\sqrt2 t)^2}$$

$$\frac{1}{\sqrt2}\left(\lim_{a\to+\infty}\int_0^{a}\frac{d(\sqrt2 t)}{1+(\sqrt2 t)^2} + \lim_{b\to-\infty}\int_b^{0}\frac{d(\sqrt2 t)}{1+(\sqrt2 t)^2}\right)$$

$$= \frac{\sqrt2}{2}\left(\lim_{a\to+\infty}\underbrace{arctg\sqrt2 a}_{\frac{\pi}{2}} - \lim_{b\to-\infty}\underbrace{arctg\sqrt2 b}_{-\frac{\pi}{2}}\right) = \frac{\sqrt2}{2}\cdot\pi$$

(11.3.)

ЗАДАЗ: ИЗРАЧУНАТИ $\displaystyle\int_0^{+\infty} \frac{x \ln x}{(1+x^2)^3}\,dx$

РЕШЕНИЕ:

$$\int_0^{+\infty} \frac{x\ln x}{(1+x^2)^3}\,dx = \lim_{\substack{a\to 0\\ b\to+\infty}} \int_a^b \frac{x\ln x}{(1+x^2)^3}\,dx = \begin{vmatrix} u=\ln x & , dv=\dfrac{x\,dx}{(1+x^2)^3} \\ du=\dfrac{dx}{x} & , v=-\dfrac{1}{4(1+x^2)^2} \end{vmatrix}$$

$$=\lim_{\substack{a\to 0\\ b\to+\infty}} \left[-\frac{\ln x}{4(1+x^2)^2}\Big|_a^b + \frac{1}{4}\int_a^b \frac{dx}{x(1+x^2)^2}\right]$$

$$\frac{1}{x(1+x^2)^2} = \frac{A}{x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2}$$

$$1 = A(1+x^2)^2 + (Bx+C)x(1+x^2) + (Dx+E)x$$

$$1 = A(1+2x^2+x^4) + (Bx^2+Cx)(1+x^2) + Dx^2+Ex$$

$$1 = A(1+2x^2+x^4) + Bx^2+Bx^4+Cx+Cx^3+Dx^2+Ex$$

$$1 = A+2Ax^2+Ax^4+Bx^2+Bx^4+Cx+Cx^3+Dx^2+Ex$$

$A+B=0$

$C=0$

$2A+B+D=0$

$C+E=0$

$A=1,\quad B=-1,\quad C=0,\quad D=-1,\quad E=0$

$$=\frac{1}{4}\lim_{\substack{a\to 0\\ b\to+\infty}}\left[-\frac{\ln b}{(1+b^2)^2}+\frac{\ln a}{(1+a^2)^2}+\int_a^b\frac{dx}{x}-\int_a^b\frac{x\,dx}{1+x^2}-\int_a^b\frac{x\,dx}{(1+x^2)^2}\right]$$

$$=\frac{1}{4}\lim_{\substack{a\to 0\\ b\to+\infty}}\left[-\frac{\ln b}{(1+b^2)^2}+\frac{\ln a}{(1+a^2)^2}+\ln x\Big|_a^b-\frac{1}{2}\ln(1+x^2)\Big|_a^b+\frac{1}{2(1+x^2)}\Big|_a^b\right]$$

$$=\frac{1}{4}\lim_{\substack{a\to 0\\ b\to+\infty}}\left[-\frac{\ln b}{(1+b^2)^2}+\frac{\ln a}{(1+a^2)^2}+\ln b-\ln a-\frac{1}{2}\ln(1+b^2)+\frac{1}{2}\ln(1+a^2)+\frac{1}{2(1+b^2)}-\frac{1}{2(1+a^2)}\right]$$

$$= \frac{1}{4}\left[\lim_{a\to 0}\left(\frac{\ln a}{(1+a^2)^2} - \ln a + \frac{1}{2}\underbrace{\ln(1+a^2)}_{0} - \underbrace{\frac{1}{2(1+a^2)}}_{\frac{1}{2}}\right)\right.$$

$$\left.+\lim_{b\to+\infty}\left(-\frac{\ln b}{(1+b^2)^2} + \ln b - \frac{1}{2}\ln(1+b^2) + \underbrace{\frac{1}{2(1+b^2)}}_{0}\right)\right]$$

$$= \frac{1}{4}\left[\lim_{a\to 0}\left(\frac{\ln a}{(1+a^2)^2} - \ln a\right) - \frac{1}{2} + \lim_{b\to+\infty}\left(-\underbrace{\frac{\ln b}{(1+b^2)^2}}_{0} + \underbrace{\ln\frac{b}{\sqrt{1+b^2}}}_{0}\right)\right] \stackrel{*}{=}$$

$\infty - \infty$

$$\lim_{a\to 0} \ln a \cdot \frac{1-(1+a^2)^2}{\underbrace{(1+a^2)^2}_{1}} = \lim_{a\to 0} \boxed{\ln a \cdot \frac{(-2a^2 - a^4)}{\underbrace{(1+a^2)^2}_{1}}}$$

$\infty \cdot 0$ $\quad -\infty \cdot 0$

$$= -\lim_{a\to 0}\frac{\ln a}{\frac{1}{\underbrace{2a^2+a^4}_{a^2(2+a^2)}}} = -\lim_{a\to 0}\frac{\frac{1}{a}}{\frac{-(4a+4a^3)}{a^4(2+a^2)^2}} = \lim_{a\to 0}\frac{a^3(2+a^2)^2}{4a\underbrace{(1+a^2)}_{1}} = 0$$

$$-\lim_{b\to+\infty}\frac{\ln b}{(1+b^2)^2} = -\lim_{b\to+\infty}\frac{\frac{1}{b}}{2(1+b^2)\cdot 2b} = 0$$

$$\lim_{b\to+\infty}\ln\underbrace{\frac{b}{\sqrt{1+b^2}}}_{\to 1} = 0$$

ūa ρè $\quad I = \frac{1}{4}\cdot\left(-\frac{1}{2}\right) = \boxed{-\frac{1}{8}}$

ЗАДАЦИ (И.3.)

ИЗРАЧУНАТИ $I=\int_0^{+\infty}\frac{\operatorname{arctg}x}{(1+x)^2}\,dx$

РЕШЕЊЕ:

$$I=\lim_{a\to+\infty}\int_0^a\frac{\operatorname{arctg}x}{(1+x)^2}\,dx=\left|\begin{array}{ll} u=\operatorname{arctg}x, & dv=\frac{dx}{(1+x)^2}\\ du=\frac{dx}{1+x^2}, & v=-\frac{1}{1+x}\end{array}\right|$$

$$=\lim_{a\to+\infty}\left[-\frac{\operatorname{arctg}x}{1+x}\Big|_0^a+\int_0^a\frac{dx}{(1+x)(1+x^2)}\right]\overset{*}{=}$$

$$\frac{1}{(1+x)(1+x^2)}=\frac{A}{1+x}+\frac{Bx+C}{1+x^2}$$

$$1=A+Ax^2+(Bx+C)(1+x)$$

$$1=A+Ax^2+Bx+Bx^2+C+Cx$$

$$\left.\begin{array}{l}A+C=1\\A+B=0\\B+C=0\end{array}\right\}-\quad\left.\begin{array}{l}A-C=0\\A+C=1\end{array}\right\}+\quad\boxed{A=\frac{1}{2}},\ \boxed{B=-\frac{1}{2}}\quad\boxed{C=\frac{1}{2}}$$

$$\int_0^a\frac{dx}{(1+x)(1+x^2)}=\frac{1}{2}\int_0^a\frac{dx}{1+x}-\frac{1}{2}\int_0^a\frac{x-1}{x^2+1}\,dx$$

$$=\frac{1}{2}\left(\ln(1+x)\Big|_0^a-\frac{1}{2}\int_0^a\frac{x\,dx}{x^2+1}+\frac{1}{2}\int_0^a\frac{dx}{x^2+1}\right)$$

$$=\frac{1}{2}\left(\ln(1+a)-\ln 1\right)-\frac{1}{2}\cdot\frac{1}{2}\ln(x^2+1)\Big|_0^a+\frac{1}{2}\operatorname{arctg}x\Big|_0^a\Big)$$

$$=\frac{1}{2}\left(\ln(1+a)-\frac{1}{4}\ln(a^2+1)+\frac{1}{2}\operatorname{arctg}a\right)$$

$$I=\lim_{a\to+\infty}\Big[\underbrace{-\frac{\operatorname{arctg}a}{1+a}}_{\to 0}+\underbrace{\frac{1}{2}\ln\frac{1+a}{\sqrt{a^2+1}}}_{\to 0}+\underbrace{\frac{1}{2}\operatorname{arctg}a}_{\to\frac{\pi}{2}}\Big)=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$