ЕКСТРЕМИ ФУНКЦИЈА ВИШЕ ПРОМЈЕНЉИВИХ

ПР1: Одредити екстреме функције

$$f(x,y) = x^4 + y^4 - 4xy \qquad , \text{(З.О.} \ \forall (x,y) \in R^2$$

Рјешење:

$$\frac{\partial f}{\partial x} = 4x^3 - 4y = 0 \quad \wedge \quad \frac{\partial f}{\partial y} = 4y^3 - 4x = 0$$

$$x^3 = y \quad \wedge \quad x^{\cancel{3}} = y^3$$

$$y^9 - y = 0 \quad \Rightarrow \quad y(y^8 - 1) = 0$$

$$y(y^4-1)(y^4+1) = 0$$

$$y(y^2-1)(y^2+1)(y^4+1) = 0$$

$$y_1 = 0, \quad y_2 = 1, \quad y_3 = -1$$

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = -1$$

$T_1(0,0)$, $T_2(1,1)$, $T_3(-1,-1)$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = -4 \quad , \quad \frac{\partial^2 f}{\partial y^2} = 12y^2$$

за $T_1(0,0)$ – седласта тачка

$A = 0, \quad B = -4, \quad C = 0 \quad , \quad AC - B^2 = 0 - 16 < 0$ – седласта тачка

за $T_2(1,1)$:

$A = 12, \quad B = -4 \quad , \quad C = 12 \quad , \quad AC - B^2 = 144 - 16 > 0$ и $A > 0$

па је тачка $T_2(1,1)$ тачка минимума

$f(1,1) = 1 + 1 - 4 = -2$

за $T_3(-1,-1)$

$A = 12, \quad B = -4, \quad C = +12 \quad , \quad AC - B^2 = 144 - 16 > 0 \quad , \quad A > 0$

$T_3(-1,-1)$ – тачка минимума

$f(-1,-1) = 1 + 1 - 4 = -2$

ЗАД 3: Одредити екстреме функције

$$f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z} \qquad (x,y,z>0)$$

Рјешење:

$$\frac{\partial f}{\partial x} = 1 - \frac{y^2}{4x^2} = 0$$

$$\frac{\partial f}{\partial y} = \frac{2y}{4x} - \frac{z^2}{y^2} = \frac{y}{2x} - \frac{z^2}{y^2} = 0$$

$$\frac{\partial f}{\partial z} = \frac{2z}{y} - \frac{2}{z^2} = 0$$

$y^2 = 4x^2 \Rightarrow y = \pm 2x$ ш. $\boxed{y = 2x}$ (због $\underbrace{x>0,\ y>0,\ z>0}$)

$\frac{y}{2x} = \frac{z^2}{y^2} \Rightarrow 1 = \frac{z^2}{y^2} \Rightarrow y^2 = z^2 \Rightarrow \boxed{y = z}$

$\frac{\cancel{2}z}{y} = \frac{\cancel{2}}{z^2} \Rightarrow y = z^3 \Rightarrow \quad z = z^3 \Rightarrow z(1-z^2) = 0$

$z>0 \Rightarrow 1 - z^2 = 0$

$\boxed{z = 1}$

$\boxed{y = 1}$

$\boxed{x = \frac{1}{2}}$

стационарна тачка $(\frac{1}{2}, 1, 1)$

$\frac{\partial^2 f}{\partial x^2} = \frac{y^2 \cdot 2\cancel{x}}{\cancel{4}x^{\cancel{4}}} = \frac{y^2}{2x^3}$, $\frac{\partial^2 f}{\partial y^2} = \frac{1}{2x} + \frac{2z^2}{y^3}$

$\frac{\partial^2 f}{\partial x \partial y} = -\frac{2y}{4x^2} = -\frac{y}{2x^2}$, $\frac{\partial^2 f}{\partial y \partial z} = -\frac{2z}{y^2}$

$\frac{\partial^2 f}{\partial x \partial z} = 0$, $\frac{\partial^2 f}{\partial z^2} = \frac{2}{y} + \frac{4}{z^3}$

$$\begin{vmatrix} \frac{1}{2\cdot\frac{1}{8}} & -\frac{1}{2\cdot\frac{1}{4}} & 0 \\ -2 & 1+2 & -2 \\ 0 & -2 & 2+4 \end{vmatrix} = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{vmatrix}$$

$\Delta_1 = 4 > 0$

$\Delta_2 = \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 12 - 4 = 8 > 0$

$\Delta_3 = 72 - 16 - 24 = 32 > 0$

$(\frac{1}{2}, 1, 1)$ тачка мин. $f(\frac{1}{2}, 1, 1) = 4$

Силвестеров критеријум:

1° $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, ... максимум

2° $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$ ... минимум.

ЗАД: Одредити екстреме функције

$$f(x,y,z) = \ln|2xy - y| + xy - x - z^2.$$

Решење:

Д.П. $2xy - y \neq 0 \iff y(2x-1) \neq 0$

$y \neq 0 \wedge x \neq \frac{1}{2}$

$$\frac{\partial f}{\partial x} = \frac{2y}{2xy-y} + y - 1 = 0, \quad \frac{2\cancel{y}}{\cancel{y}(2x-1)} + y - 1 = 0$$

$$\boxed{\frac{2}{2x-1} + y - 1 = 0} \quad (1)$$

$$\frac{\partial f}{\partial y} = \frac{2x-1}{2xy-y} + x = 0 \Rightarrow \frac{\cancel{2x-1}}{y\cancel{(2x-1)}} + x = 0$$

$$\boxed{\frac{1}{y} + x = 0} \quad (2)$$

$$\frac{\partial f}{\partial z} = -z = 0 \Rightarrow \boxed{z = 0}$$

$$\boxed{\frac{2}{2x-1} - 1 = y}, \quad x = -\frac{1}{y} \Rightarrow \boxed{y = -\frac{1}{x}}$$

$$\cancel{\frac{2}{2x-1} - 1 = -\frac{1}{x}}$$

$$\cancel{\frac{\overbrace{2 - 2x + 1}^{3-2x}}{2x-1} = -\frac{1}{x}} \Rightarrow \cancel{(3-2x)\cdot x = -2x+1}$$

$$\cancel{3x - 2x^2 + 2x - 1 = 0}$$

$$\cancel{-2x^2 + 5x - 1 = 0}$$

$$2xy - y - 2x + 3 = 0, \quad x = -\frac{1}{y}$$

$$\underbrace{y^2 - y - 2 = 0}, \quad \underbrace{y_1 = -1, \quad y_2 = 2}_{x_1 = 1, \quad x_2 = -\frac{1}{2}}$$

$S_1(1,-1,0)$, $S_2(-\frac{1}{2}, 2, 0)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{4}{(2x-1)^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = 1, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial y \partial z} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{y^2}, \quad \frac{\partial^2 f}{\partial z^2} = -2$$

$$d^2 f(1,-1,0) = -4dx^2 - dy^2 - 2dz^2 + 2dxdy$$
$$= -3dx^2 - dx^2 - dy^2 - 2dz^2 + 2dxdy$$
$$\boxed{d^2 f = -3dx^2 - (dx - dy)^2 - 2dz^2 < 0}$$

за $S_1(1,-1,0)$

$$\begin{vmatrix} -4 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix}, \quad \Delta_1 = -4 < 0, \quad \Delta_2 = 3 > 0, \quad \Delta_3 = -6 < 0$$

па у S_1 имамо максимум

$$\underline{f(1,-1,0) = -2}$$

за $S_2(-\frac{1}{2}, 2, 0)$

$$\begin{vmatrix} -1 & 1 & 0 \\ 1 & -1/4 & 0 \\ 0 & 0 & -2 \end{vmatrix} \qquad \Delta_1 = -1 < 0, \quad \Delta_2 = -\frac{3}{4} < 0, \quad \Delta_3 = \frac{3}{2} > 0$$

S_2 није тачка екстрема

$$d^2 f(-\tfrac{1}{2}, 2, 0) = -dx^2 - \frac{1}{4}dy^2 - 2dz^2 + 2dxdy$$
$$= -(dx - dy)^2 + \frac{3}{4}dy^2 - 2dz^2$$

– може бити и позитивно и негативно

ЗАД: Одредити екстреме функције

$$f(x,y) = \sin x + \sin y + \sin(x+y) \quad , \quad (x,y \in [0, 2\pi])$$

РЈЕШЕЊЕ:

(1)-(2)

$$\begin{rcases} \frac{\partial f}{\partial x} = \cos x + \cos(x+y) = 0 \\ \frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0 \end{rcases} \Rightarrow \begin{matrix} \cos\frac{y}{2} \cdot \cos\frac{2x+y}{2} = 0 \\ \cos\frac{x}{2} \cos\frac{x+2y}{2} = 0 \end{matrix}$$

трансформиш (1)-(2)

$$\cos\frac{y}{2} = 0 \quad \vee \quad \cos\frac{2x+y}{2} = 0$$

$$\cos\frac{x}{2} = 0 \quad \vee \quad \cos\frac{x+2y}{2} = 0$$

$$\frac{y}{2} = \frac{\pi}{2} \Rightarrow y = \pi$$

$$\frac{x}{2} = \frac{\pi}{2} \Rightarrow x = \pi \quad , \quad \boxed{T_1(\pi, \pi)}$$

$$\frac{2x+y}{2} = \frac{\pi}{2} \Rightarrow 2x + y = \pi \Rightarrow y = \pi - 2x$$

$$\frac{x+2y}{2} = \frac{\pi}{2} \Rightarrow x + 2y = \pi \Rightarrow x + 2\pi - 4x = \pi \Rightarrow \boxed{x = \frac{\pi}{3}}$$

$$y = \pi - \frac{2\pi}{3} = \boxed{\frac{\pi}{3}}$$

$$\boxed{T_2\left(\frac{\pi}{3}, \frac{\pi}{3}\right)}$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

$$\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

за $T_1(\pi, \pi)$

$A = 0$, $B = 0$, $C = 0$ – не можемо ништа рећи за екстрем

Тражимо први наредни диференцијал III и вишег реда који је различит од нуле.

$$\frac{\partial^3 f}{\partial x^3}(x,y) = -\cos x - \cos(x+y)$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = -\cos(x+y)$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = -\cos(x+y)$$

$$\frac{\partial^3 f}{\partial y^3} = -\cos y - \cos(x+y)$$

$$d^3 f(\pi,\pi) = (1-1)dx^3 + (1-1)dy^3 + 3(-1)dx^2dy - 3dx\,dy^2$$
$$= -3dx^2dy - 3dx\,dy^2$$

није сталног предзнака, што је особина диференцијала непарног реда.

Предзнак од $d^3 f(\pi,\pi)$ може бити и позитиван и негативан, што зависи од прираштаја $dx = x-\pi$ и $dy = y-\pi$, па ни израз (који се добија развојем функције по Тејлоровој формули)

$$f(x,y) - f(\pi,\pi) + df(\pi,\pi) + \frac{1}{2!}d^2f(\pi,\pi) + \frac{1}{3!}d^3f(\pi+\theta(x-\pi), \pi+\theta(y-\pi))$$

није сталног предзнака ни у једној околини тачке (π,π), па функција f нема екстрем у тачки (π,π) (него седласту тачку).

$T_2(\frac{\pi}{3},\frac{\pi}{3})$, $A = -\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} = -\sqrt{3}$

$B = -\sin\frac{2\pi}{3} = -\frac{\sqrt{3}}{2}$, $C = -\sqrt{3}$

$AC - B^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$, $A < 0$ у тачки $(\frac{\pi}{3},\frac{\pi}{3})$ макс.

$$f_{max}\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

Зад 1: Одредити екстреме функције

$z = x + y + 4\sin x \sin y$

Решење:

$$\frac{\partial z}{\partial x} = 1 + 4\cos x \sin y = 0 \Rightarrow 1 - 2\sin(x-y) + 2\sin(x+y) = 0$$

$$\frac{\partial z}{\partial y} = 1 + 4\sin x \cos y = 0 \Rightarrow 1 + 2\sin(x-y) + 2\sin(x+y) = 0$$

$$2 + 4\sin(x+y) = 0 \Rightarrow \boxed{\sin(x+y) = -\frac{1}{2}} \quad (1)$$

$$\cancel{1} + 2\sin(x-y) - \cancel{1} = 0 \Rightarrow \boxed{\sin(x-y) = 0} \quad (2)$$

$$(1) \Rightarrow x + y = (-1)^{k+1}\frac{\pi}{6} + k\pi, \quad k \in \mathbb{Z}$$

$$(2) \Rightarrow x - y = s\pi, \quad s \in \mathbb{Z}$$

$$2x = (-1)^{k+1}\frac{\pi}{6} + k\pi + s\pi \Rightarrow \boxed{x = (-1)^{k+1}\frac{\pi}{12} + (k+s)\frac{\pi}{2}}$$

$$y = (-1)^{k+1}\frac{\pi}{6} + k\pi - (-1)^{k+1}\frac{\pi}{12} - (k+s)\frac{\pi}{2}$$

$$\boxed{y = (-1)^{k+1}\frac{\pi}{12} + (k-s)\frac{\pi}{2}}$$

$$\frac{\partial^2 z}{\partial x^2} = -4\sin x \sin y, \quad \frac{\partial^2 z}{\partial x \partial y} = 4\cos x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -4\sin x \sin y$$

$$\boxed{AC - B^2} = 16\sin^2 x \sin^2 y - 16\cos^2 x \cos^2 y = -16\cos(x-y)\cos(x+y)$$

$$\Delta = -16\underbrace{\cos(s\pi)}_{(-1)^s}\cdot\cos\left((-1)^{k+1}\frac{\pi}{6}+k\pi\right)$$

$$\cancel{=+16(-1)} = -16\cdot(-1)^s\cos\left((-1)^{k+1}\frac{\pi}{6}+k\pi\right)$$

$$= 16\cdot(-1)^{s+1}\left(\cos(-1)^{k+1}\frac{\pi}{6}\cdot\underbrace{\cos k\pi}_{(-1)^k} - \underbrace{\sin k\pi}_{0}\cdot\sin(-1)^{k+1}\frac{\pi}{6}\right)$$

$$= \boxed{16(-1)^{s+k+1}\cdot\cos\frac{\pi}{6}}$$

За $s+k+1$ – парно $\Delta>0$ и има екстрем

а ако је $s+k+1$ – непарно $\Delta<0$ нема екстрем

Значи да функција има екстрем за $s+k$ – непарно, а у том случају су бројеви

или s и k различите парности

За испитивање карактера екстрема

$$\frac{\partial^2 z}{\partial x^2} = 2\cos(x+y) - 2\cos(x-y)$$

$$= 2\left(\cos\left((-1)^{k+1}\frac{\pi}{6}+k\pi\right)-\cos(s\pi)\right)$$

$$= 2\left(\cos(-1)^{k+1}\cdot\frac{\pi}{6}\cdot\cos k\pi + 0 - \cancel{\cos}(-1)^s\right)$$

$$= 2\left(\frac{\sqrt{3}}{2}(-1)^k - (-1)^s\right) = \sqrt{3}(-1)^k - 2(-1)^s$$

$(k=2m)$ $(s=2m-1)$

ако је k – парно, онда је s – непарно

$\sqrt{3}+2 = A$ па функција има минимум

а обрнуто $-\sqrt{3}-\sqrt{2} = A$ и функција има максимум

$z_{min} = 2m\pi - 2 - \sqrt{3} - \frac{\pi}{6}$, $m \in Z$

$z_{max} = (2m-1)\pi + 2 + \sqrt{3} + \frac{\pi}{6}$, $m \in Z$

6.3:

Зад 2: Наћи условни екстремум функције

$u(x,y,z) = x^2 + 2y^2 - z^2 + 2xy - 3xz + yz + x$

на правој $p: \begin{cases} x + y + z = 0 \\ 2x - y + 2z = 0 \end{cases}$

Решење:

$\left.\begin{array}{l} x + y + z = 0 \\ 2x - y + 2z = 0 \end{array}\right\} +$ $\quad 3x + 3z = 0$

$\boxed{x = -z}$

пресјечна тачка $(-z, 0, z)$

$u(z) = \cancel{z^2} - \cancel{z^2} + 3z^2 - z$

$\frac{\partial u}{\partial z} = 6z - 1 = 0 \Rightarrow z = \frac{1}{6}$

$\frac{\partial^2 u}{\partial z^2} = 6$, мин $\left(-\frac{1}{6}, 0, \frac{1}{6}\right)$

$z\left(-\frac{1}{6}, 0, \frac{1}{6}\right) = \cancel{\frac{1}{36}} - \cancel{\frac{1}{36}} + 3 \cdot \frac{1}{36} - \frac{1}{6} = \frac{1}{12} - \frac{1}{6} = \frac{1-2}{12}$

$z\left(-\frac{1}{6}, 0, \frac{1}{6}\right) = -\frac{1}{12}$

условни екстреми

ЗАД 1: Одредити екстреме функције
$u = x^2+2y^2+3z^2$ на лопти $x^2+y^2+z^2 \le 100$.

Решење:

$F(x,y,z,\lambda) = x^2+2y^2+3z^2+\lambda(x^2+y^2+z^2-100)$

$\frac{\partial F}{\partial x} = 2x+\lambda x = 0 \Rightarrow 2x(1+\lambda)=0 \Rightarrow x=0 \vee \lambda=-1$

$\frac{\partial F}{\partial y} = 4y+2\lambda y = 0 \Rightarrow 2y\cdot(2+\lambda)=0 \Rightarrow y=0 \vee \lambda=-2$

$\frac{\partial F}{\partial z} = 6z+2\lambda z = 0 \Rightarrow 2z(3+\lambda)=0 \Rightarrow z=0 \vee \lambda=-3$

Не може бити $x=y=z=0$ због $x^2+y^2+z^2=100$.

$x \neq 0 \Rightarrow \underline{\lambda=-1}$

$2y\cdot 1=0 \Rightarrow y=0$

$2z\cdot 2=0 \Rightarrow z=0$

$x^2=100 \Rightarrow x=\pm 10 \quad M_{1,2}(\pm 10,0,0)$

$y \neq 0 \Rightarrow \lambda=-2$

$2x\cdot(-1)=0 \Rightarrow x=0$

$2z\cdot 1=0 \Rightarrow z=0$

$y^2=100 \Rightarrow y_{3,4}=\pm 10, \quad M_{3,4}(0,\pm 10,0)$

$z \neq 0 \Rightarrow \lambda=-3 \Rightarrow x=0, \ y=0, \ z=\pm 10$

$M_{5,6}(0,0,\pm 10)$

$\frac{\partial^2 F}{\partial x^2} = 2+2\lambda, \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial x \partial z} = \frac{\partial^2 F}{\partial y \partial z} = 0$

$\frac{\partial^2 F}{\partial y^2} = 4+2\lambda$

$\frac{\partial^2 F}{\partial z^2} = 6+2\lambda$

$$d^2F = (2+2\lambda)dx^2 + (4+2\lambda)dy^2 + (6+2\lambda)dz^2$$

$\underline{\lambda = -1}$, $M_{1,2}(\pm 10, 0, 0)$

$d^2F = 2dy^2 + 4dz^2 > 0$ па су тачке $M_{1,2}$ тачке минимума.

$\underline{\lambda = -2}$, $M_{3,4}(0, \pm 10, 0)$

$d^2F = -2dx^2 + 2dz^2$ – променљивог знака седласте тачке

$\underline{\lambda = -3}$, $M_{5,6}(0, 0, \pm 10)$

$d^2F = -4dx^2 - 2dy^2 < 0$ па су $M_{5,6}$ тачке максимума

$u_{мин}(\pm 10, 0, 0) = 100$

$u_{макс}(0, 0, \pm 10) = 300$

ЗАД 2: Наћи условне екстреме функције $u = x^2+y^2-xz$ на кривој датој са $xy+2=0$, $x+y-z-8=0$.

РЈЕШЕЊЕ:

$$F(x,y,z,\lambda_1,\lambda_2) = x^2+y^2-xz+\lambda_1(xy+2)+\lambda_2(x+y-z-8)$$

$$\frac{\partial F}{\partial x} = 2x - z + \lambda_1 y + \lambda_2 = 0 \quad (1)$$

$$\frac{\partial F}{\partial y} = 2y + \lambda_1 x + \lambda_2 = 0 \quad (2)$$

$$\frac{\partial F}{\partial z} = -x - \lambda_2 = 0 \quad (3) \Rightarrow \boxed{\lambda_2 = -x}$$

(4) $xy+2=0 \Rightarrow xy=-2 \Rightarrow \boxed{x = -\frac{2}{y}}$, $y \neq 0$

(5) $x+y-z-8=0 \Rightarrow \boxed{z = x+y-8}$

$(1) \Rightarrow \cancel{2x} - \cancel{x} - y + 8 + \lambda_1 y - \cancel{x} = 0 \Rightarrow \boxed{8 - y + \lambda_1 y = 0}$

$$\Downarrow$$

$(2) \Rightarrow 2y + \lambda_1 x - x = 0$

$\boxed{\lambda_1 = \frac{8-y}{-y}}$, $y \neq 0$

уврстимо у 2)

$$2y + \frac{8-y}{-y}\cdot\left(-\frac{2}{y}\right) + \frac{2}{y} = 0 \quad / \cdot \frac{y^2}{2}$$

$\cancel{\frac{y^3 - 16 + 2y}{y} +}$

$$y^3 + 8 - \cancel{y} + \cancel{y} = 0$$

$$y^3 + 8 = 0 \Rightarrow (y+2)(y^2-2y+4) = 0$$

$\boxed{y=-2}$ $\boxed{x=1}$

$\boxed{z=-9}$

$T(1,-2,-9)$

$$\frac{\partial^2 F}{\partial x^2} = 2, \quad \frac{\partial^2 F}{\partial y^2} = 2, \quad \frac{\partial^2 F}{\partial z^2} = 0$$

$$\frac{\partial^2 F}{\partial x \partial y} = \lambda_1, \quad \frac{\partial^2 F}{\partial x \partial z} = -1, \quad \frac{\partial^2 F}{\partial y \partial z} = 0$$

$$d^2F = 2dx^2 + 2dy^2 + 2\lambda_1 dxdy - 2dxdz$$

$$\lambda_1 = \frac{8-y}{-y} = \frac{8+2}{2} = 5$$

На основу (5) $x+y-z-8=0 \Rightarrow dx+dy-dz=0$

$$\underline{dz = dx+dy}$$

$$d^2F = 2\left(dx^2+dy^2+5dxdy - dx(dx+dy)\right)$$

$$d^2F = 2\left(\cancel{dx^2}+dy^2+5dxdy - \cancel{dx^2} - dxdy\right)$$

$$d^2F = 2\left(dy^2+4dxdy\right)$$

(4) $xy+2=0 \Rightarrow ydx+xdy=0$

$$-2dx+dy=0 \Rightarrow dy = 2dx$$

$$\boxed{dx = \frac{dy}{2}}$$

$$d^2F = 2\left(dy^2+4\cdot\frac{dy}{2}\cdot dy\right) = 2(dy^2+2dy^2) = 6dy^2 > 0$$

па је тачка $T(1,-2,-9)$ тачка минимума функције.

$$u(1,-2,-9) = 1+4+9 = 14$$

ЗАД 3: Нађи минимално растојање тачке $A(1,0)$ до елипсе $4x^2+9y^2=36$.

РЈЕШЕЊЕ:

$M(x,y)$ – тачка са елипсе

$$d^2(A,M) = (x-1)^2+y^2$$

$$F(x,y,\lambda) = (x-1)^2+y^2+\lambda(4x^2+9y^2-36)$$

$$\frac{\partial F}{\partial x} = 2(x-1)+8\lambda x = 0 \Rightarrow x-1+4\lambda x=0$$

$$\frac{\partial F}{\partial y} = 2y+18\lambda y=0 \Rightarrow y+9\lambda y=0$$

$$x(1+4\lambda)=1 \Rightarrow x=\frac{1}{1+4\lambda}$$

$$y(1+9\lambda)=0 \Rightarrow y=0 \vee \lambda=-\frac{1}{9}$$

$$\boxed{y=0} \Rightarrow 4x^2=36, \quad x^2=9, \quad x=\pm 3$$

$$T_{1,2}(\pm 3, 0)$$

$$d^2(A,T_1)=(3-1)^2=4 \Rightarrow d(A,T_1)=\sqrt{4}=2$$

$$d^2(A,T_2)=(-3-1)^2=16 \Rightarrow d(A,T_2)=4$$

$$\boxed{\lambda=-\frac{1}{9}}, \quad x=\frac{1}{1+4\lambda}=\frac{1}{1-\frac{4}{9}}=\frac{9}{5}$$

$$4\cdot\frac{81}{25}+9y^2=36 \Rightarrow 9y^2=36-\frac{4\cdot 81}{25} \quad /:9$$

$$y^2=4-\frac{36}{25}=\frac{64}{25}, \quad \boxed{y=\pm\frac{8}{5}}$$

$\boxed{T_{3,4}\left(\frac{9}{5},\pm\frac{8}{5}\right) \text{ - најближе тачки } A}$

$$d^2(A,T_{3,4})=\left(\frac{9}{5}-1\right)^2+\frac{64}{25}=\frac{16}{25}+\frac{64}{25}=\frac{80}{25}; \quad d(A,T_{3,4})=\frac{4\sqrt{5}}{5}=\frac{4}{\sqrt{5}}$$

Зад 4: На елипсоиду $x^2+2y^2+3z^2=1$ наћи тачку која је најближа и тачку која је најдаља од равни $x+2y-3z=0$.

Рјешење:

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

$$F(x,y,z,\lambda)=x+2y-3z+\lambda(x^2+2y^2+3z^2-1)$$

$$\frac{\partial F}{\partial x}=1+2\lambda x=0 \Rightarrow x=-\frac{1}{2\lambda}$$

$$\frac{\partial F}{\partial y}=2+4\lambda y=0 \Rightarrow y=-\frac{1}{2\lambda}$$

$$\frac{\partial F}{\partial z}=-3+6\lambda z=0 \Rightarrow z=\frac{1}{2\lambda}$$

$$\frac{1}{4\lambda^2}+\frac{2}{4\lambda^2}+\frac{3}{4\lambda^2}=1 \Rightarrow 6=4\lambda^2 \Rightarrow \lambda=\pm\sqrt{\frac{3}{2}}$$

$$\lambda=\sqrt{\frac{3}{2}},\quad M_1\left(-\frac{1}{\sqrt6},-\frac{1}{\sqrt6},\frac{1}{\sqrt6}\right)$$

$$\lambda=-\sqrt{\frac{3}{2}},\quad M_2\left(\frac{1}{\sqrt6},\frac{1}{\sqrt6},-\frac{1}{\sqrt6}\right)$$

$$d_{M_1}=\frac{\left|-\frac{1}{\sqrt6}-\frac{2}{\sqrt6}-\frac{3}{\sqrt6}\right|}{\sqrt{14}}=\frac{\frac{6}{\sqrt6}}{\sqrt{14}}=\frac{6}{\sqrt{6\cdot14}}=\frac{3}{\sqrt{21}}$$

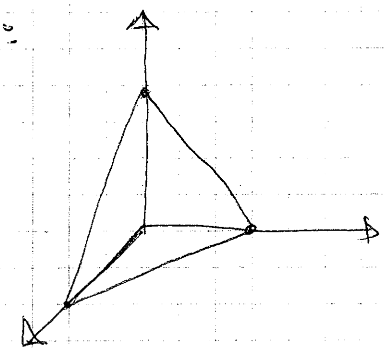
$$d_{M_2}=\frac{\left|\frac{1}{\sqrt6}+\frac{2}{\sqrt6}+\frac{3}{\sqrt6}\right|}{\sqrt{14}}=\frac{3}{\sqrt{21}}$$

Како се раван и елипсоид сијеку (јер пролазе кроз координатни почетак) то су тачке M_1 и M_2 најдаље од равни (симетричне)

ЗАД 5: (2.07.2005.)

Одредити екстреме функције $u = xy + yz + zx - 2xyz$ на троуглу $x+y+z=1$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

РЕШЕЊЕ:



У унутрашњости троугла

$$\frac{\partial u}{\partial x} = y + z - 2yz = 0 \quad (1)$$

$$\frac{\partial u}{\partial y} = x + z - 2xz = 0 \quad (2)$$

$$\frac{\partial u}{\partial z} = y + x - 2xy = 0 \quad (3)$$

$(1)-(2) \Rightarrow y - x - 2yz + 2xz = 0$

$y - x - 2z(y-x) = 0 \Rightarrow \boxed{(y-x)(1-2z) = 0}$

$y - x = 0 \lor 1 - 2z = 0$

$(1)-(3) \Rightarrow z - x - 2yz + 2xy \Rightarrow \boxed{(z-x)(1-2y) = 0}$

$\boxed{z - x = 0 \lor 1 - 2y = 0}$

$y = x, \quad z = x$ тј. $x = y = z$

$3x = 1 \Rightarrow \boxed{x = \frac{1}{3}}$

$T_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \in$ унутр. троугла

$y = x, \quad y = \frac{1}{2}, \quad x = \frac{1}{2}; \quad \frac{1}{2} + \frac{1}{2} + z = 1 \Rightarrow z = 0$

$T_2\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ - на рубу

$z = x \Rightarrow z = \frac{1}{2} \Rightarrow y = 0$, $(\frac{1}{2}, 0, \frac{1}{2})$ – руб

$(0, \frac{1}{2}, \frac{1}{2})$ – руб.

На рубу су и тачке $(0,0,1)$, $(1,0,0)$, $(0,1,0)$

$u(0,0,1) = u(0,1,0) = u(1,0,0) = 0$ – мин.

$u(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{7}{27}$ – макс

$u(0, \frac{1}{2}, \frac{1}{2}) = u(\frac{1}{2}, 0, \frac{1}{2}) = u(\frac{1}{2}, \frac{1}{2}, 0) = \frac{1}{4}$

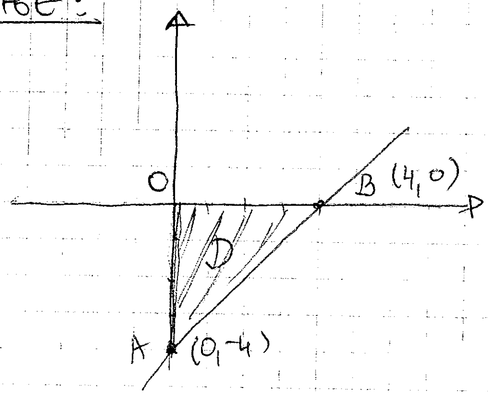
ЗАД 5: (22.06.2009.)

Одредити највећу и најмању вриједност функције

$z = 2x^2 + xy + y^2 - 7x$ на области

$D = \{(x,y): x \geq 0,\ 0 \geq y \geq x-4\}$

РЈЕШЕЊЕ:



У унутрашности

$$\frac{\partial z}{\partial x} = 4x + y - 7 = 0$$

$$\frac{\partial z}{\partial y} = x + 2y = 0$$

$4x + y = 7$

$x + 2y = 0 \Rightarrow x = -2y \Rightarrow -8y + y = 7 \Rightarrow \boxed{\begin{matrix} y = -1 \\ x = 2 \end{matrix}}$

$T_1(2,-1)$ је стационарна тачка из D

$\boxed{f(2,-1) = -7}$

На рубу:

$\overline{OB}$: $y = 0$, $z = f(x,0) = 2x^2 - 7x$

$f'_x = 4x - 7 = 0 \Rightarrow x = \frac{7}{4}$

тј. $T_2(\frac{7}{4}, 0)$, $f(\frac{7}{4}, 0) = -\frac{49}{8} \approx -6{,}125$

$\overline{OA}$: $x = 0$

$f(0,y) = y^2$, $f'_y = 2y$, $y = 0$

$T_3(0,0)$, $f(0,0) = 0$

$\overline{AB}$: $x = x$, $y = x - 4$

$f(x, x-4) = 2x^2 + x(x-4) + (x-4)^2 - 7x = 4x^2 - 19x + 16$

$f'_x = 8x - 19 = 0$ $x = \frac{19}{8}$, $y = \frac{19}{8} - 4 = -\frac{13}{8}$

$T_4\left(\frac{19}{8}, -\frac{13}{8}\right)$

$f\left(\frac{19}{8}, -\frac{13}{8}\right) \approx -6,565$

у теменима $(4,0)$, $(0,-4)$, $f(0,4) = 4$

$f(0,-4) = 16$

па функција има минимум у тачки $(2,-1)$

$f_{мин}(2,-1) = -7$, а максимум у тачки $(0,-4)$

тј. $f_{макс}(0,-4) = 16$

ЗАД 1: Одредити екстреме функције

$u = xy + yz$, под условима $x^2+y^2=2$; $y+z=2$

$(x>0, y>0, z>0)$

Решење:

$$F(x,y,z,\lambda_1,\lambda_2) = xy + yz + \lambda_1(x^2+y^2-2)+\lambda_2(y+z-2)$$

$$\frac{\partial F}{\partial x} = y + 2\lambda_1 x = 0 \quad (1)$$

$$\frac{\partial F}{\partial y} = x + z + 2\lambda_1 y + \lambda_2 = 0 \quad (2)$$

$$\frac{\partial F}{\partial z} = y + \lambda_2 = 0 \quad (3) \qquad \lambda_2 = -y$$

$z = 2-y$, $\quad x^2+y^2=2$

$y + 2\lambda_1 x = 0 \Rightarrow 2\lambda_1 x = -y$, $\quad \lambda_1 = -\frac{y}{2x}$

$x + 2 - y + 2\lambda_1 y - y = 0$

$x + 2 - 2y + 2\cdot\left(-\frac{y}{2x}\right)\cdot y = 0$

$x + 2 - 2y - \frac{y^2}{x} = 0 \quad /\cdot x$

$$\left.\begin{array}{l} x^2+2x-2xy-y^2=0 \\ x^2+y^2=2 \end{array}\right| +$$

$2x^2+2x-2xy = 2 \quad /:2$

$x^2+x-xy-1=0$

$$\boxed{\frac{x^2+x-1}{x} = y}$$

$$x^2 + \frac{(x^2+x-1)^2}{x^2} = 2$$

$$x^4 + (x^2+x)^2 - 2(x^2+x) + 1 = 2x^2$$

$$x^4 + x^4 + 2x^3 + x^2 - 2x^2 - 2x + 1 - 2x^2 = 0$$

$$\underline{2x^4 + 2x^3 - 3x^2 - 2x + 1 = 0}$$

$\pm 1,$ $\quad 2+2-3-2+1=0$

$$\begin{array}{c|ccccc} 1 & 2 & 2 & -3 & -2 & 1 \\ \hline & 2 & 4 & 1 & -1 & 0 \end{array}$$

$\boxed{x_1 = 1}$ $\qquad 2x^3 + 4x^2 + x - 1 = 0$

$\qquad -2 + 4 - 1 - 1 = 0$

$\boxed{x_2 = -1}$

$$\begin{array}{c|cccc} -1 & 2 & 4 & 1 & -1 \\ \hline & 2 & 2 & -1 & 0 \end{array}$$

$$2x^2 + 2x - 1 = 0$$

$$x_{3,4} = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4}$$

$$x_3 = \frac{-1+\sqrt{3}}{2}, \quad x_4 = \frac{-1-\sqrt{3}}{2}$$

због $x \geq 0$

узимамо $x_1 = 1$, $\quad x_3 = \frac{-1+\sqrt{3}}{2}$

$$y = \frac{x^2 + x - 1}{x} = \frac{2x^2+2x-2}{2x} = \frac{1-2}{2x} = \frac{-1}{\cancel{2} \cdot \frac{-1+\sqrt{3}}{\cancel{2}}}$$

$$= \frac{-1}{-1+\sqrt{3}} = \frac{-1}{0{,}73} < 0$$ та остаје

само $\boxed{x_1 = 1}$

$$y = \frac{1+x-x}{1} = 1, \quad z = 2 - y = 2-1 = 1$$

$$(1,1,1), \quad \lambda_2 = -1, \quad \lambda_1 = -\frac{1}{2}$$

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda_1, \quad \frac{\partial^2 F}{\partial x \partial y} = 1, \quad \frac{\partial^2 F}{\partial x \partial z} = 0$$

$$\frac{\partial^2 F}{\partial y^2} = 2\lambda_1, \quad \frac{\partial^2 F}{\partial y \partial z} = 1, \quad \frac{\partial^2 F}{\partial z^2} = 0$$

$$d^2F = 2\lambda_1 dx^2 + 2\lambda_1 dy^2 + 2dxdy + 2dydz$$

$$= -2dx^2 - 2dy^2 + 2dxdy + 2dydz$$

$$dy + dz = 0 \Rightarrow \boxed{dz = -dy}$$

$$d^2F = -2dx^2 - 2dy^2 + 2dxdy - 2dy^2$$

$$2xdx + 2ydy = 0 \qquad x=1,\ y=1$$

$$\boxed{dx = -dy} \qquad dy = -dx$$

$$-6$$

$$d^2F = -2dx^2 - 2dx^2 - 2dx^2 - 2dx^2 = -8dx^2 < 0$$

значи у тачки (1,1,1) има условни максимум $\boxed{u_{max} = 2}$

ЗАД 2: Одредити највећу и најмању вредност функције $f(x,y,z)=x^2+y^2+z^2$ под условом $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ $(a>b>c>0)$

Решење:

$$F(x,y,z,\lambda)=x^2+y^2+z^2+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\frac{\partial F}{\partial x}=2x+\frac{2\lambda x}{a^2}=0 \Rightarrow x\left(1+\frac{\lambda}{a^2}\right)=0$$

$$\frac{\partial F}{\partial y}=2y+\frac{2\lambda y}{b^2}=0 \Rightarrow y\left(1+\frac{\lambda}{b^2}\right)=0$$

$$\frac{\partial F}{\partial z}=2z+\frac{2\lambda z}{c^2}=0 \Rightarrow z\left(1+\frac{\lambda}{c^2}\right)=0$$

$x=0 \lor 1+\frac{\lambda}{a^2}=0 \Rightarrow \lambda=-a^2$

$x=0 \land y=0$, $z=\pm c$, $\lambda=-c^2$

$x=0 \land z=0$, $y=\pm b$, $\lambda=-b^2$

$x=\pm a$, $y=0$, $z=0$, $\lambda=-a^2$

$$\frac{\partial^2 F}{\partial x^2}=2+\frac{2\lambda}{a^2},\quad \frac{\partial^2 F}{\partial x\partial y}=0=\frac{\partial^2 F}{\partial x\partial z}=\frac{\partial^2 F}{\partial y\partial z}$$

$$\frac{\partial^2 F}{\partial y^2}=2+\frac{2\lambda}{b^2},\quad \frac{\partial^2 F}{\partial z^2}=2+\frac{2\lambda}{c^2}$$

$$d^2F=2\left(1+\frac{\lambda}{a^2}\right)dx^2+2\left(1+\frac{\lambda}{b^2}\right)dy^2+2\left(1+\frac{\lambda}{c^2}\right)dz^2$$

$$(0,0,\pm c) \Rightarrow d^2F=2\underbrace{\left(1-\frac{c^2}{a^2}\right)}_{>0}dx^2+2\underbrace{\left(1-\frac{c^2}{b^2}\right)}_{>0}dy^2$$

због $a^2>b^2>c^2$

па у тим тачкама има минимум

$f_{min} = c^2$

$(0, \pm b, 0)$, $\lambda = -b^2$

$d^2F = 2(1 - \frac{b^2}{a^2})dx^2 + 2(1 - \frac{b^2}{c^2})dz^2$ нема екстрем

$\underbrace{a^2 - b^2}_{>0}$, $\underbrace{c^2 - b^2}_{<0}$

$(\pm a, 0, 0)$, $t = -a^2$

$d^2F = 2(1 - \frac{a^2}{b^2})dy^2 + 2(1 - \frac{a^2}{c^2})dz^2 < 0$

$b^2 - a^2 < 0$, $c^2 - a^2 < 0$

па у тим тачкама има максимум

$f_{max} = a^2$

ИСПИТНИ ЗАДАЦИ ИЗ ЕКСТРЕМА

1) Одредити условне екстреме функције

$u = xyz$ при условима

$x^2+y^2+z^2=1$, $x+y+z=0$

Решење:

$F(x,y,z,\lambda_1,\lambda_2) = xyz + \lambda_1(x^2+y^2+z^2-1) + \lambda_2(x+y+z)$

$\frac{\partial F}{\partial x} = yz + 2\lambda_1 x + \lambda_2 = 0 \Rightarrow \lambda_2 = -yz - 2\lambda_1 x$

$\frac{\partial F}{\partial y} = xz + 2\lambda_1 y + \lambda_2 = 0 \Rightarrow xz + 2\lambda_1 y - yz - 2\lambda_1 x = 0$ (1)

$\frac{\partial F}{\partial z} = xy + 2\lambda_1 z + \lambda_2 = 0 \Rightarrow xy + 2\lambda_1 z - yz - 2\lambda_1 x = 0$ (2)

(1) $\Rightarrow z(x-y) + 2\lambda_1(y-x) = 0 \Rightarrow (x-y)(z-2\lambda_1) = 0$

(2) $\Rightarrow y(x-z) + 2\lambda_1(z-x) = 0 \Rightarrow (x-z)(y-2\lambda_1) = 0$

$x=y \lor z=2\lambda_1$

$x=z \lor y=2\lambda_1$

$x=y=z=0$ - немогуће због $x^2+y^2+z^2=1$

$y=x=2\lambda_1$

$y+x+z=0 \Rightarrow 2x+z=0 \Rightarrow \boxed{z=-4\lambda_1}$

из $x^2+y^2+z^2=1 \Rightarrow 4\lambda_1^2+4\lambda_1^2+16\lambda_1^2=1 \Rightarrow 24\lambda_1^2=1$

$\lambda_1^2 = \frac{1}{24} \Rightarrow \lambda_1 = \pm\frac{1}{2\sqrt{6}}$

$x=y=\pm\frac{1}{\sqrt{6}}$, $z=\mp\frac{2}{\sqrt{6}}$

због симетрије $x=y \wedge x=z$

$T_1\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), T_2\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), T_3\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$T_4\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), T_5\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), T_6\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

$\frac{\partial^2 F}{\partial x^2} = 2\lambda_1 \quad , \frac{\partial^2 F}{\partial x \partial y} = z \quad , \frac{\partial^2 F}{\partial x \partial z} = y$

$\frac{\partial^2 F}{\partial y^2} = 2\lambda_1 \quad , \frac{\partial^2 F}{\partial y \partial z} = x$

$\frac{\partial^2 F}{\partial z^2} = 2\lambda_1$

$$d^2F = \frac{\partial^2 F}{\partial x^2}dx^2 + \frac{\partial^2 F}{\partial y^2}dy^2 + \frac{\partial^2 F}{\partial z^2}dz^2 + 2\frac{\partial^2 F}{\partial x \partial y}dxdy + 2\frac{\partial^2 F}{\partial y \partial z}dydz + 2\frac{\partial^2 F}{\partial x \partial z}dxdz$$

$dx+dy+dz=0$

$\underline{dz=-(dx+dy)}$

$\wedge \quad 2xdx+2ydy+2zdz=0$

$xdx+ydy+zdz=0$

како је $x=y=2\lambda_1 \ , \ z=-4\lambda_1$

$\underline{2\lambda_1 dx+2\lambda_1 dy-4\lambda_1 dz=0}$

$2\lambda_1 dx+2\lambda_1 dy+4\lambda_1(dx+dy)=0$

$6\lambda_1 dx+6\lambda_1 dy=0 \quad /:6\lambda_1$

$\boxed{dx+dy=0}$

$$d^2F = 2\lambda_1(dx^2+dy^2+dz^2)+2zdxdy+2xdydz+2ydxdz$$

$$d^2F = 2\lambda_1(dx^2+dy^2+dz^2)+2zdxdy+2xdy(-dx-dy)+2ydx(-dx-dy)$$

$$d^2F = 2\lambda_1(dx^2+dy^2+dz^2)+2zdxdy-2xdxdy-2xdy^2-2ydx^2-2ydxdy$$

$$d^2F = 2\lambda_1(dx^2+dy^2+dz^2)-8\lambda_1 dxdy-4\lambda_1 dxdy-4\lambda_1 dy^2-4\lambda_1 dx^2-4\lambda_1 dxdy$$

$$= 2\lambda_1(dx^2+dy^2+dz^2)-8\lambda_1 dxdy-4\lambda_1 dxdy-4\lambda_1 dy^2-4\lambda_1 dx^2-4\lambda_1 dxdy$$

$$d^2F = 2\lambda_1(dx^2+dy^2+dz^2) - 16\lambda_1 dx dy - 4\lambda_1 d^2y^2 - 4\lambda_1 d^2x^2$$

$$d^2F = 2\lambda_1[dx^2+dy^2+dz^2 - 8dxdy - 2(d^2x^2+d^2y^2)]$$

$$d^2F = 2\lambda_1[-dx^2-dy^2+dz^2 - 8dxdy]$$

$$d^2F = 2\lambda_1[-dx^2-dy^2+(dx+dy)^2 - 8dxdy]$$

$$d^2F = 2\lambda_1[-dx^2-dy^2+dx^2+2dxdy+dy^2 - 8dxdy]$$

$$d^2F = 2\lambda_1[-\cancel{dx^2}-\cancel{dy^2}+\cancel{dx^2}+2dxdy+\cancel{dy^2} - 8dxdy]$$

$$d^2F = 2\lambda_1[-6dxdy] = -12\lambda_1 dxdy$$

$\lambda_1 = \frac{1}{2\sqrt{6}} \Rightarrow d^2F < 0 \Rightarrow T_1, T_2, T_3$ су условне макс.

$\Rightarrow T_4, T_5, T_6$ -//- мин.

$\lambda_2 = -\frac{1}{2\sqrt{6}} \Rightarrow d^2F > 0$

2) Одредити условне екстреме функције

$u = xy + yz$, при условима

$x^2+y^2=2$, $y+z=2$, $x>0$, $y>0$

Решење:

$$F(x,y,z,\lambda_1,\lambda_2) = xy + yz + \lambda_1(x^2+y^2-2) + \lambda_2(y+z-2)$$

$$\frac{\partial F}{\partial x} = y + 2\lambda_1 x = 0 \quad \Rightarrow \quad \boxed{y = -2\lambda_1 x}$$

$$\frac{\partial F}{\partial y} = x+z+2\lambda_1 y+\lambda_2 = 0$$

$$\frac{\partial F}{\partial z} = y + \lambda_2 = 0 \Rightarrow \boxed{\lambda_2 = -y}$$

$$\left.\begin{aligned} x+z+2\lambda_1 y - \overset{\lambda_2}{y} &= 0 \\ z &= 2-y \end{aligned}\right\} \Rightarrow x+2-y+2\lambda_1 y - y = 0$$

$$x+2-2y+2\lambda_1 y = 0$$

$$x+2+4\lambda_1 x - 4\lambda_1^2 x = 0$$

$$x(1+4\lambda_1-4\lambda_1^2) = -2$$

$$\boxed{x = \frac{2}{4\lambda_1^2 - 4\lambda - 1}}$$

$$\boxed{y = \frac{-4\lambda_1}{4\lambda_1^2-4\lambda-1}}$$

$$\frac{4}{(4\lambda_1^2-4\lambda_1-1)^2} + \frac{16\lambda_1^2}{(4\lambda_1^2-4\lambda_1-1)^2} = 2$$

$$4 + 16\lambda_1^2 = 2(4\lambda_1^2-4\lambda_1-1)^2$$

$$\boxed{16\lambda_1^4 - 32\lambda_1^3 + 8\lambda_1 - 1 = 0}$$

$$\boxed{\lambda_1 = \frac{1}{2} \ \vee\ \lambda_1 = -\frac{1}{2}}$$

$$\lambda_1 = \frac{2+\sqrt{3}}{2}, \quad \lambda_1 = \frac{2-\sqrt{3}}{2}$$

$\lambda_1 = \frac{1}{2} \Rightarrow x < 0$ није рјешење

$\lambda_1 = -\frac{1}{2} \Rightarrow \boxed{x = 1} \quad \boxed{y = z = 1}$ $\qquad (1, 1, 1)$

$\lambda_1 = \frac{2+\sqrt{3}}{2} \Rightarrow x = -2 < 0$ - није рј.

$\lambda_1 = \frac{2-\sqrt{3}}{2} \Rightarrow x < 0$

$(1, 1, 1)$

$\frac{\partial^2 F}{\partial x^2} = 2\lambda_1, \quad \frac{\partial^2 F}{\partial x \partial y} = 1, \quad \frac{\partial^2 F}{\partial x \partial z} = 0$

$\frac{\partial^2 F}{\partial y^2} = 2\lambda_1, \quad \frac{\partial^2 F}{\partial y \partial z} = 1, \quad \frac{\partial^2 F}{\partial z^2} = 0$

$d^2F = 2\lambda_1 dx^2 + 2\lambda_1 dy^2 + 2dxdy + 2dydz$

$y + z = 2 \Rightarrow dy + dz = 0$

$x^2 + y^2 = 2 \Rightarrow 2xdx + 2ydy = 0$

$d^2F = -dx^2 - dy^2 + 2dxdy + 2dy(-dy)$

$= -(dx - dy)^2 - 2dy^2 < 0$

$(1, 1, 1)$ максимум

3) Нaћи условни екстремум функције утврђен на вј.

$u(x,y,z) = x^2+2y^2-z^2+2xy-3xz+yz+x$

на правој

$p: \begin{cases} x+y+z=0 \\ 2x-y+2z=0 \end{cases} \overset{+}{\Rightarrow} \quad 3x+3z=0 \Rightarrow \underline{x=-z}$

пресечна тачка $(-z,0,z)$

$u(z) = z^2 - z^2 + 3z^2 - z$

$\frac{\partial u}{\partial z} = 6z-1 \Rightarrow z=\frac{1}{6}$ $\quad (-\frac{1}{6}, 0, \frac{1}{6})$ мин.

$\frac{\partial^2 u}{\partial z^2} = 6$

4) На елипсоиду $\frac{x^2}{96}+y^2+z^2=1$. Наћи тачку која је најмање (највише) удаљена од равни

$3x+4y+12z=288$

Решење:

$F(x,y,z,\lambda) = 3x+4y+12z-288+\lambda\left(\frac{x^2}{96}+y^2+z^2-1\right)$

$\frac{\partial F}{\partial x} = 3+\frac{2\lambda x}{96}=0 \Rightarrow x=-\frac{144}{\lambda}$

$\frac{\partial F}{\partial y} = 4+2\lambda y=0 \quad y=-\frac{2}{\lambda}$

$\frac{\partial F}{\partial z} = 12+2\lambda z=0 \quad z=-\frac{6}{\lambda}$

$\frac{(144)^2}{96\lambda^2}+\frac{4}{\lambda^2}+\frac{36}{\lambda^2}=1 \Rightarrow \boxed{\lambda=\pm 16}$

за $\boxed{\lambda=16} \Rightarrow x=-9,\ y=-\frac{1}{8},\ z=-\frac{3}{8}$

$T_1(-9,-\frac{1}{8},-\frac{3}{8})$

$\boxed{\lambda=-16}, \quad T_2(9,\frac{1}{8},\frac{3}{8})$

$T_1 \left(-9, -\frac{1}{8}, \frac{-3}{8}\right)$

$$\begin{vmatrix} \frac{1}{39} & 0 & 0 \\ 0 & \frac{32}{13} & 0 \\ 0 & 0 & \frac{32}{13} \end{vmatrix}$$

$D_1 > 0$, $D_2 > 0$, $D_3 > 0$

T_1 – је минимум

$T_2 \left(9, \frac{1}{8}, \frac{3}{8}\right)$ $\lambda = -16$

$$\begin{vmatrix} -\frac{1}{39} & 0 & 0 \\ 0 & -\frac{32}{13} & 0 \\ 0 & 0 & -\frac{32}{13} \end{vmatrix}$$

$D_1 < 0$

$D_2 > 0$

$D_3 < 0$

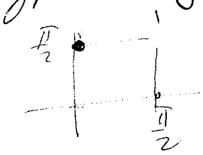
T_2 – је максимум

ЗАД: (6.07.2009)

Одредити екстреме функције

$f(x,y) = \cos x + \cos y + \sin(x+y)$ на квадрату

$\{(x,y) : 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}\}$



Решење:

$$\frac{\partial f}{\partial x} = -\sin x + \cos(x+y) = 0 \quad \Rightarrow \quad \sin x = \sin y, \quad x,y \in [0, \tfrac{\pi}{2}]$$

$$\frac{\partial f}{\partial y} = -\sin y + \cos(x+y) = 0 \quad \Rightarrow \quad \underline{\underline{x = y}}$$

$$-\sin x + \cos 2x = 0$$
$$-\sin x + \cos^2 x - \sin^2 x = 0$$
$$-\sin x + 1 - 2\sin^2 x = 0$$
$$-2\sin^2 x - \sin x + 1 = 0, \quad 2\sin^2 x + \sin x - 1 = 0$$
$$(2\sin x - 1)(\sin x + 1) = 0$$

$$\sin x = \frac{1}{2} \quad \vee \quad \sin x = -1$$

$$x = \frac{\pi}{6} \quad \vee \quad x = \frac{3\pi}{2} \notin \text{датом квадрату}$$

$$y = \frac{\pi}{6}$$

стационарна тачка је $(\frac{\pi}{6}, \frac{\pi}{6})$

$$f(\tfrac{\pi}{6}, \tfrac{\pi}{6}) = \frac{3\sqrt{3}}{2} \approx 2{,}6$$

Екстреми на рубу:

$$f(x,0) = \cos x + 1 + \sin x, \quad x \in [0, \tfrac{\pi}{2}]$$

$$f'(x,0) = -\sin x + \cos x = 0 \Rightarrow \sin x = \cos x \Rightarrow x = \frac{\pi}{4}$$

$$f(\tfrac{\pi}{4}, 0) = \sqrt{2} + 1 \approx 2{,}41$$

$$f(x, \tfrac{\pi}{2}) = \cos x + \sin(x + \tfrac{\pi}{2}) = \cos x + \cos x = 2\cos x$$

$$f'_x(x, \tfrac{\pi}{2}) = 2\sin x = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0, \quad x = \pi \notin [0, \tfrac{\pi}{2}]$$

$$\boxed{f(0, \tfrac{\pi}{2}) = 2}$$

$x=0, \quad y \in [0, \frac{\pi}{2}]$

$f(0,y) = \cos y + \sin y + 1$

$f'_y = -\sin y + \cos y = 0 \qquad \sin y = \cos y \quad \Rightarrow \quad y = \frac{\pi}{4}$

$\underline{f(0, \frac{\pi}{4}) = \sqrt{2}+1}$

$x = \frac{\pi}{2}, \quad y \in [0, \frac{\pi}{2})$

$f(\frac{\pi}{2}, y) = \cos y + \sin(y + \frac{\pi}{2}) = \cos y + \cos y = 2\cos y$

$f'_y = \sin y = 0 \quad \Rightarrow \quad y=0$ и $y = \pi$

$\boxed{f(\frac{\pi}{2}, 0) = 2}$

$f(\frac{\pi}{2}, \frac{\pi}{2}) = 0 \qquad , \qquad f(0,0) = 1+1 = 2$

па функција има минимум у тачки $(\frac{\pi}{2}, \frac{\pi}{2})$

Одг. $f_{min} = f(\frac{\pi}{2}, \frac{\pi}{2}) = 0$

$f_{max} = f(\frac{\pi}{6}, \frac{\pi}{6}) = \frac{3\sqrt{3}}{2}$

ЗАД 1: ОДРЕДИТИ ЕКСТРЕМЕ ФУНКЦИЈЕ

$f(x,y) = \sin x + \sin y + \sin(x+y)$, $(x,y \in [0,2\pi])$

РЈЕШЕЊЕ:

$\frac{\partial f}{\partial x} = \cos x + \cos(x+y) = 0 \Rightarrow \cos\frac{y}{2}\cdot\cos\frac{2x+y}{2} = 0 \Rightarrow$

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0 \Rightarrow \cos\frac{x}{2}\cdot\cos\frac{x+2y}{2} = 0 \Rightarrow$

$\cos\frac{y}{2} = 0 \lor \cos\frac{2x+y}{2} = 0 \Rightarrow \frac{y}{2} = \frac{\pi}{2} \Rightarrow \underline{y = \pi} \lor \underline{2x+y = \pi}$

$\cos\frac{x}{2} = 0 \lor \cos\frac{x+2y}{2} = 0 \Rightarrow x = \pi \lor x+2y = \pi$

(π,π) , $\left.\begin{array}{l} 2x+y = \pi \quad /(-2) \\ x+2y = \pi \end{array}\right\} +$ $\left.\begin{array}{l} -4x-2y = -2\pi \\ x+2y = \pi \end{array}\right\} + \Rightarrow \boxed{\begin{array}{l} x = \frac{\pi}{3} \\ y = \frac{\pi}{3} \end{array}}$

(π,π) , $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

$\frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$, $\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

ЗА (π,π)

$A = B = C = 0$, $\Delta = 0$

$\frac{\partial^3 f}{\partial x^3} = -\cos x - \cos(x+y)$, $\frac{\partial^2 f}{\partial x^2 \partial y} = -\cos(x+y)$

$\frac{\partial^3 f}{\partial^2 y \partial x} = -\cos(x+y)$, $\frac{\partial^2 f}{\partial y^3} = -\cos y - \cos(x+y)$

$d^3 f(x,y) = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^3 f(x,y) = \frac{\partial^3 f(x,y)}{\partial x^3} d^3x + 3\frac{\partial^3 f}{\partial x^2 \partial y} d^2x\,dy$

$+ 3\frac{\partial^3 f}{\partial x \partial^2 y} dx\,d^2y + \frac{\partial^2 f}{\partial y^3} d^3y \Rightarrow$

$d^3 f(\pi,\pi) = 3(-1)d^2x\,dy - 3\,dx\,dy^2 = -3dx\,dy(dx+dy)$

$= -3(x-\pi)(y-\pi)(x-\pi+y-\pi) = -3(x-\pi)(y-\pi)(x+y-2\pi) \neq 0$

Како $d^3f(\pi,\pi)$ није сталног предзнака, што је иначе особина диференцијала непарног реда, предзнак од $d^3f(\pi,\pi)$ може бити и позитиван и негативан што зависи од прираштаја $dx = x-\pi$, $dy = y-\pi$, то ни израз (који се добија развојем функције по Тејлоровој формули)

$$f(x,y) = f(\pi,\pi) + df(\pi,\pi) + \frac{1}{2!} d^2 f(\pi,\pi) + \frac{1}{3!} d^3 f(\pi+\theta(x-\pi), \pi+\theta(y-\pi))$$

није сталног предзнака ни у једној околини тачке (π,π). Зато закључујемо (на основу дефиниције екстремума) да функција f има у тачки (π,π) седло тј. да је седласта тачка.

у $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$, $A = C = -\sqrt{3} < 0$, $B = -\frac{\sqrt{3}}{2}$

$$AC - B^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

$$f_{max} = f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$$